Private–Information Insurance Contracts in Continuous Time

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Abstract

A continuous–time private–information insurance contract is analyzed. The special case analyzed is equivalent to a credit equilibrium. Defection from the contract and its equivalent credit equilibrium using smooth–pasting methods is analyzed. The contract and its equivalent standard credit equilibrium cannot be made immune to defection. Immunity from defection can be reinstated by constructing a constrained insurance contract as described in [3], but the insurance value provided is reduced below that of the full–information case.
PRIVATE-INFORMATION INSURANCE CONTRACTS
IN CONTINUOUS TIME
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Abstract. A continuous-time private-information insurance contract is analyzed. The special case analyzed is equivalent to a credit equilibrium. Defection from the contract and its equivalent credit equilibrium using smooth-pasting methods is analyzed. The contract and its equivalent standard credit equilibrium cannot be made immune to defection if there is insufficient patience. Immunity from defection can be reinstated by constructing a constrained insurance contract as described in [4], but the insurance value provided is reduced below that of the full-information case.

1. INTRODUCTION
Consider a collection of individuals whose endowments evolve stochastically and independently from each other. If they are risk averse, the individuals could initially confer and agree to pool and then evenly redistribute their endowments through the entire future. If there are enough individuals the pooled endowment will be constant or nearly constant. All the risk arising from the endowment fluctuations would be eliminated. Such a contract would be the ideal of an income or wealth redistribution program.

Individuals would see no difference between this income redistribution contract and an insurance scheme in which the insurance provider simply offset all endowment fluctuations, positive or negative. This is the perspective that will be taken in this paper. Two constraints will be imposed on such a contract. The first is a private information constraint: it will be assumed that individuals can observe their endowments at each instant, but that the insurance provider, which will be referred to as the principal, will be unable to directly observe the endowment. Instead, the principal relies on the individual's reports and uses the history of the individual's signals and of consumption, to construct an incentive-compatible contract.

The construction of the contract emulates that of [6] and [2] or more recently [5] in the sense that the principal views the utility of the agent as a control variable and the discounted value of utility as a state variable, except that the endowment process evolves in continuous time. Continuous-time methods are easier to use than discrete-time methods in some ways, particularly with regard to the statement and use of incentive-compatibility conditions: here they are simply first and second order conditions.

As in the literature on such contracts there is a resemblance of the contract's consumption allocation to that of a credit equilibrium. As in [14], [15], when a condition is imposed requiring that the principal have zero expected profit, the equivalence is exact. It is therefore possible to analyze the standard credit equilibrium from the standpoint of insurance theory. It is not surprising that the necessity of imposing incentive constraints on the model results in impaired insurance, and the corresponding credit equilibrium also displays this impairment. It is tempting to label this as inefficient or second best, but it is more appropriately identified as constrained-efficient, the constraint arising from the scarcity of information.

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When interpreting the contract as a redistribution scheme within a large population of individuals, the central consequence of the informational constraint is that wealth and its attendant consumption grows increasingly disparate with the passage of time. Nevertheless, the contract is ex-ante efficient. As has already been noted in the literature on dynamic contracts with private information, incentive compatibility and equality of income are incompatible. What is new here is the observation that asset equilibria of the kind that seem to prevail empirically reflect this fact.

With the credit equilibrium expressed as a kind of dynamic game, it is natural to ask if the game is robust with respect to noncooperative behavior; this is the second constraint on the contract that will be explored. Because both the contract and the lending equilibrium are demonstrated to operate as a risk-sharing mechanism, it is a requirement of equilibrium that individuals with good luck remain within the system, or the redistribution that effects the insurance will be infeasible. If high-wealth individuals renege, or in game-theoretic parlance, defect, then the contract and the equivalent credit equilibrium are not feasible, because the consumption of low-endowment individuals cannot be funded. The fact that wealth disparity grows over time in the private information contract suggests that pressure to defect in this sense will grow commensurately with wealth.

If such defection can be avoided, the contract can be decentralized: the contract then is subgame perfect and can be taken as a description of actual economies. However, the temptation to defect and the attendant breakdown of feasibility does inevitably occur in the standard contract and credit model if agents lack sufficient patience. This finding suggests that standard asset models also suffer from this breakdown.

Salvaging the equilibrium requires that the punishments of defectors be modified. It is not obvious that a successful modification exists, and whether the modification leaves intact the asset-like features of the contract. It is demonstrated here that such a modification does indeed exist and that the asset-like properties of the contract are preserved. The additional structure of the model does however require background institutions that resemble equity arrangements more than credit arrangements. These background institutions are necessarily fragmented in a way that results in impairment of the insurance beyond the impairment arising from the private information. ¹

Because credit and equity are both present, an equity premium is generated. Unlike the equity premium generated in standard representative-agent models, this premium reflects the informational constraints in the model.

In the following section the basic private-information contract is developed. The technical focus is a derivation which permits the principal's problem to be expressed using the individual's discounted utility value as the state variable. The main features of the contract for a CARA example are then developed. The equivalence of this contract to a credit equilibrium is then demonstrated. In the final part of the paper, the modifications of the contract necessitated by the requirement of nondefection are developed.

¹ A similar finding is in [14].
2. THE CONTRACT

There is a principal and an agent. The agent, who is risk-averse, has an endowment process \( y(t) \) that is risky. The principal is risk-neutral and provides insurance. The principal cannot observe the agent’s endowment process however, and must therefore impose incentive constraints on the agent in order to elicit truthful signals about the endowment.

The situation will be represented by a kind of dual problem in which the principal treats the agent’s utility as a control, the agent’s discounted value of utility as a state, and minimizes a cost function whose argument is the agent’s utility. If the agent is given utility \( u \) at instant \( t \), the principal will have to provide the consumption \( C(u) \) that finances it. Thus, the cost function \( C \) is just \( u^{-1} \) if \( u \) is utility. The principal’s objective is thus

\[
\min_{\{u(t)\}} E \int_0^\infty e^{-rt} C(u(t)) dt \quad u(0) \text{ given}
\]  

(2.1)

The notation used here is deliberately redolent of the [2] notation. \(^2\)

The agent’s problem is as follows. There is an endowment process \( y(t) \) that is a Brownian motion:

\[
dy(t) = \sigma dz(t) \tag{2.2}
\]

There is a state variable \( x(t) \) that can be thought of as endowment if the principal provides no insurance. If the principal does provide insurance, then the state variable is artificially constructed by the principal to elicit information from the agent. The state variable evolves as a diffusion process:

\[
dx(t) = f(x, \zeta)dt + g(x)d\zeta(t) \tag{2.4}
\]

where \( \zeta \) is the signal sent by the agent to the principal about the utility he should currently receive. The drift function \( f \) is assumed differentiable.

The state \( x \) is separate, and for the moment \( x \) is left as an abstract process loosely related to endowment. The signal could also be represented as the consumption process he desires based on his endowment, but since utility is an invertible function of consumption it is simpler to assume that utility is reported. Thus, \( \zeta = u(c(t)) \). (And, as in [2], \( c(t) = C(\zeta) \) in the equilibrium contract.) The agent thus solves

\[
\max_{\zeta(t)} E \int_0^\infty e^{-rt} \zeta(x(t)) dt \tag{2.5}
\]

subject to (2.4), and for notational consistency the principal’s objective is

\[
\min_{\{\zeta(t)\}} E \int_0^\infty e^{-rt} C(\zeta(t)) dt \tag{2.1'}
\]

Although the signal is of utility, if it is measurable with respect to the agent’s endowment process, then it is appropriate to think of \( \zeta \) also as a signal sent by the agent about the

\(^2\) The conditioning filtration has been notationally suppressed; the principal must choose utility that is measurable with respect to the filtration generated by the agent’s signal and of consumption. Incentive compatibility will ultimately force this filtration to be equivalent to the filtration generated by the endowment.
true state $y$, the endowment. Whether the measurability requirement is satisfied will be verified later.

The principal chooses the drift function $f(\cdot, \cdot)$ in order to capture the history of the signals the agent has sent, which will then be an input into the utility functional.

**Contract constraints.** In the dynamic contract literature there are two constraints that apply in private information settings: incentive and promise-keeping. I will refer to the promise keeping constraint as the continuation constraint here.

Incentive constraints simply express the optimization by the agent in the face of physical and contractual constraints. In the literature these constraints often result in corner solutions, but in the present setting interior optima exist. The agent’s optimization can therefore be represented by a standard Bellman equation.

The continuation constraint is similarly tractable: the differentiability of the contract structure allows the continuation constraint to be expressed using Ito’s lemma.

The agent’s value process. The expression of the incentive constraint begins with the Bellman equation. The agent chooses a signal process, solving (2.5) subject to (2.4). The resulting Bellman equation is:

$$
\rho V(x) = \max_{\zeta} \{ \zeta + f(x, \zeta)V'(x) + \frac{1}{2}g(x)^2V''(x) \} \quad (2.6)
$$

The optimality condition is

$$
1 + f_{\zeta}V' = 0 \quad (2.7)
$$

Observe that the optimality condition expresses a connection between two functions, $f$ and $V$, which are as yet undetermined.

Combining the optimality condition (2.7), the state equation (2.4) and the Bellman equation (2.6) yields the condition

$$
\rho V(x) = \{ \zeta^*(x, V) + f(x, \zeta^*)V'(x) + \frac{1}{2}g(x)^2V''(x) \} \quad (2.6')
$$

The continuation value constraint. The continuation constraint can be obtained by first assuming the agent’s value $V$ is a function of $x(t)$, and then applying Ito’s lemma, which yields

$$
dV(t) = (f(x, u)V'(x) + \frac{1}{2}g(x)^2V''(x)) dt + g(x)V'(x)d\xi(t) \quad (2.8)
$$

It is key that the value state $V$ is viewed in current value form, so that it is independent of time. The equation expresses the law of motion for the value state, implicitly capturing the discounted future payoffs under the contract, as expressed by the state variable $x(t)$ and by the function $f$. The driving stochastic process is represented as $\xi(t)$ instead of $z(t)$ to allow for the possibility that $z(t)$ might not be measurable by the principal.
A combined differential constraint.} The optimized incentive equation (2.6′) can be combined with the continuation equation (2.8). The result is
\[
dV(t) = (\rho V - \zeta)dt + g(x)V'(x)d\xi(t)
\] (2.9)
This not only has the \(\zeta\)-variable as the control, it has the \(\rho V\) term.

The principal’s problem. The \(x\)-state is not quite the value state, \(V\), that would be the principal’s state in a discrete-time contract model. But it seems intuitively clear that there will be a one-to-one relationship between \(x\) and \(V\), although this must be later verified. Thus, it will be assumed
\[
x = \phi(V)
\]
with \(\phi\) invertible, differentiable, and monotone. In that case (2.9) can be written
\[
dV(t) = (\rho V - \zeta)dt + m(V)d\xi(t)
\] (2.10)
where \(m(V) \equiv g(\phi(V))V'(\phi(V))\).

The model is now cast very much in the Atkeson-Lucas mold, with the principal’s problem stated in terms of the cost of controlling utility subject to incentive compatibility (the agent’s optimality condition) and value continuation (the agent’s Bellman equation) jointly expressed in (2.10). The Bellman equation for the principal can now be stated:
\[
rW = \min_{\zeta} \left\{ C(\zeta) + (\rho V - \zeta)W'(V) + \frac{1}{2}m(V)^2W''(V) \right\}
\] (2.11)
with optimality condition
\[
C'(\zeta) = W'(V)
\] (2.12)
This is a standard envelope condition.

Duality.} It is straightforward to demonstrate that the following duality relationship holds:
\[
C'(u) = \frac{1}{u'(c)}
\]
where \(u\) is the agent’s utility function. The inverse relationship shows up in agency models such as [13].

This duality relationship suggests a way to explain the nature of the state variable \(x(t)\): it is identical to \(W(t)\), the principal’s value state! Some intuition for this comes from [14], in which the principal’s value turns out to be equivalent to the asset holdings of the agent for appropriate values of the discount rate \(r\). This idea is best explored through an example in the present setting.

The principal’s law of motion.} It will be useful to state the stochastic process for the principal’s value state \(W(t)\) as well as for the agent’s value state, because it will be
equivalent to the agent’s law of motion for the state $x$. Using Ito’s lemma yields the stochastic differential equation for $W$:

$$dW(t) = (-uW'(V) + \frac{1}{2}m(V)^2W''(V))dt + m(V)W'(V)d\xi(t)$$  \hspace{1cm} (2.13)$$

As with the agent’s problem, it is key that the term $-rW$ does not appear in this equation; the current value function, rather than the present value function, is being used.

Combining this equation with the principal’s Bellman equation (2.11) yields

$$dW = (rW - C(\zeta))dt + m(V)W'(V)d\bar{z}(t)$$  \hspace{1cm} (2.14)$$

This equation will be used in the example. In addition to verifying that the equation is equivalent to the law of motion for the agent’s state variable $x$ (if $m(V)W'(V)$ is constant, it has the exact form of that budget constraint for an individual in the credit equilibrium that will be set out later), the measurability of $\bar{z}(t)$ and $x(t)$ with respect to $z(t)$ and the endowment process $y(t)$ will be confirmed.

3. An Example

The example is a translation of the exponential-log example of [2]. The agent has exponential utility:

$$u(c) = -e^{-ac}$$  \hspace{1cm} (3.1)$$

The cost of providing utility is the inverse function:

$$C(u) = -\frac{1}{a} \ln(-u)$$  \hspace{1cm} (3.2)$$

It is obvious that the principal’s value function will be logarithmic and this conjecture will now be verified. The value will take the following form:

$$W(V) = A \ln(-V) + B$$  \hspace{1cm} (3.3)$$

with $A$ and $B$ constant. Using the optimality condition (2.12) to calculate $u$ in terms of $V$ yields

$$\zeta = -\frac{1}{aA}V$$  \hspace{1cm} (3.4)$$

When substituted into the principal’s Bellman equation (2.11) the conjectured form works, with the exact solution

$$W(V) = -\frac{1}{ar} \ln(-V) + \frac{1}{ar} (1 - \frac{\rho}{r} - \ln(r)) + \frac{1}{2} \frac{\mu^2}{r}$$

using the conjecture that that $m(V)$ is linear: $m(V) = -\mu V$. This conjecture is verified below.
Again using the optimality condition (2.12), the principal’s cost is

\[ C(\zeta) = rW - \frac{1}{a}(1 - \frac{\rho}{r} + \frac{1}{2}\frac{\mu^2}{r}) \]

The stochastic differential equation (2.14) becomes

\[ dW(t) = \frac{1}{a}(1 - \frac{\rho}{r} + \frac{1}{2}\frac{\mu^2}{r})dt - m(V)\frac{1}{ar}\frac{1}{V}d\zeta(t) \quad (3.5) \]

**Linear m(V).** Again using the conjecture that the function \( m(V) \) is linear,

\[ m(V) = -\mu V \]

the law of motion for the principal’s state becomes:

\[ dW(t) = \frac{1}{a}(1 - \frac{\rho}{r} + \frac{1}{2}\frac{\mu^2}{r})dt + \frac{\mu}{ar}d\zeta(t) \quad (3.6) \]

**Incentive compatibility in the example.** The principal implicitly controls the accumulation of value \( V \) by the agent via the state equation for \( x \), and in particular through the structure of the drift function \( f \).

The next step is to let the state \( x \) be equal to \( W \). In that case, the function \( \phi(V) \) (see equation (2.10) and the surrounding discussion) is just the principal’s value function \( W \). In that case, the function \( f \) must be the drift coefficient of (2.14), and similarly for the volatility coefficient:

\[ dW = f(W, \zeta)dt + g(W)d\zeta(t) \quad (3.7) \]

The coefficients can be matched to those in (2.14):

\[ f(W, \zeta) = rW - C(\zeta) = rW + (1/a)\ln(-\zeta) \]

and

\[ f_\zeta = -\frac{1}{a\zeta} \]

Using the equivalence of \( x \) and \( W \), the principal’s value function can be inverted:

\[ V(W) = -\frac{1}{r}e^{1-\frac{\phi}{2}+\frac{\mu^2}{2r}}e^{-aw} = Ke^{-aw}; \quad V'(W) = ar e^{1-\frac{\phi}{2}+\frac{\mu^2}{2r}}e^{-aw} = ar Ke^{-aw} \]

The optimality condition is then

\[ 1 = \frac{1}{\zeta}rKe^{-aw} \]
The optimal signal can be found from this:

\[ \zeta = -e^{1-\frac{\psi}{a} + \frac{1}{2} \frac{\mu^2}{r}} e^{-\alpha r W} \]  

(3.9)

Recalling the assumption that \( \zeta = u \) and the functional form for \( u \) yields the corresponding consumption:

\[ c = r W - \frac{1}{a} \left( 1 - \frac{\rho}{r} + \frac{1}{2} \frac{\mu^2}{r} \right) \]

Viewing the principal’s subjective rate of time discount \( r \) as a kind of interest rate, consumption is thus a kind of rent on the principal’s value, with an adjustment term. Re-examining (3.6), it is evident that the principal’s value is a diffusion, and therefore consumption will also be a diffusion; the ideal of perfect insurance or of perfect redistribution or even of bounded distribution cannot be attained.

Feasibility. In an equilibrium where the principal uses no resources for the contract on average—equivalent to the market clearing requirement of a credit equilibrium—consumption must be driftless. That happens if the constant term is zero: the principal’s value state process \( W \) will then also be driftless since the drift term in the \( dW \) equation will just be \( r W - r W = 0 \), and the principal’s value will have expectation of its initial state \( W(0) \). This condition is in turn satisfied if

\[ 1 - \frac{\rho}{r} + \frac{1}{2} \frac{\mu^2}{r} = 0 \]  

(3.10)

This condition will again arise in the analysis of the lending equilibrium.

Solution for \( \mu \). If \( g(x) \) is a constant \( \gamma \), then we can solve for \( m(V) \) as well:

\[ m(V) = -g(W) \frac{1}{f_\zeta} = \frac{\gamma}{C'(\zeta)} = \frac{\gamma}{W'(V)} = \gamma \alpha r V \]

which has the linear form conjectured in (3.6)! Substituting in (3.6) yields the zero-drift condition

\[ 1 - \frac{\rho}{r} + \frac{1}{2} \frac{(\gamma \alpha r)^2}{r} = 0 \]

This is the similar condition necessary for market clearing in a credit equilibrium, as will be demonstrated below.

Measurability. The measurability of \( \bar{z} \) with respect to \( z \) can now be noted. The process for the signal \( \zeta \) is given in (3.9): this process is a measurable function of \( W(t) \). In turn, \( W(t) \) is a function of \( \bar{z} \); see equation (3.6). The drift term in (3.6) consists of known parameters. Thus if \( \bar{z}(t) = z(t) \) the \( W \) process is also measurable with respect to \( z(t) \).

By way of contrast, this would not be the case if there were an unknown parameter in the drift process that had to be estimated, analogous to the bandit problem analyzed by [3].

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Temporary incentive compatibility and persistence. The remaining issue pertains to the persistence of the endowment process. In standard dynamic contract models of this kind, the endowment process is i.i.d. That allows a direct determination of the increment to endowment. Here, endowment persists. If an agent lies about his current endowment, he can keep the gains from the lie for a long time. In particular, if he understates his endowment, he can borrow, increase his consumption, and then remain on this lower perceived track of endowment forever. In the i.i.d. case, he can't gain in this fashion since his future endowment is unrelated to his present endowment.

This possibility would seem to violate the conditions for temporary incentive compatibility in the sense of Green [6], as pointed out by Phelan and Fernandes [12]. A redeeming assumption is needed. The redeeming assumption that will be made is that past consumption (or equivalently utility) is observable, even though endowment is not. This assumption is compatible with the i.i.d. literature in the sense that correct reports of endowment necessarily reveal utility and consumption there.

With this assumption in hand we are back to the situation in which it is the innovation in endowment that is reported. If the agent overreports, he must contribute a fraction of his endowment to the planner, which is clearly suboptimal for him. If he under-reports, his consumption will be subsidized, but it is revealed that his consumption is excessive once he adds his hidden endowment and this will lower his future expected utility. His consumption will thus be lower than what it would otherwise be.

Now it is apparent that if the initial state of endowment is known, the agent will report truthfully from then on; all that is needed is the initial value of utility. Does it then make sense for the agent to at least under-report his initial endowment? In the example presented here there are no wealth effects, and so even if endowment were initially understated, the incentive properties of the incremental reports would be unaffected. The agent would secretly consume a constant excess income, say $y_p$, forever, but the residual endowment process $y(t) - y_p$ would fluctuate exactly as true endowment except for that excess constant. True utility and its attendant value state would be multiplied by a constant $e^{-a y_p}$, and the incentive properties of the process are unaffected. This differs from the Phelan and Fernandes model, in which endowment is a Markov process, and preferences are more general. Insurance there is constricted by the need to account for agents misreporting their initial state, but the limited possibilities of the Markov process bounds the misreports of agents, thus rescuing at least some insurance.

4. THE EQUIVALENT CREDIT EQUILIBRIUM

In a credit equilibrium individuals choose consumption, which controls their wealth (the direct analogue of $W$) subject to perturbations by a stochastic process $z(t)$. The following is an exposition of the solution of the individual's optimal consumption and portfolio problem. It will then be demonstrated how equilibrium resource constraints determine the interest rate. Finally, the isometry between the private-information contract and the credit equilibrium will be noted.

There is a riskless bond and risky endowment. The individual must choose between
investing in the bond and consuming. The individual solves
\[
\max -E \int_0^\infty e^{-\rho t} e^{-a t} dt
\]
subject to
\[
dW(t) = (rW - c) dt + \sigma dz
\]
where z is the endowment process.

The Bellman equation is
\[
\rho V(W) = \max_c \{ -e^{-ac} + (rW - c)V'(W) + \frac{1}{2} \sigma^2 V''(W) \} \tag{4.1}
\]
The optimality condition is
\[
0 = ae^{-ac} - V'.
\]
Posit that \(V(W) = Be^{AW}\). Then the optimality condition yields
\[
c = -(A/a)W - (1/a) \ln(AB/a).
\]
Substituting in the Bellman equation yields
\[
\rho Be^{AW} = -e^{[AW + \ln(AB/a)]} + (rW + (A/a)W + (1/a) \ln(AB/a))ABe^{AW} + \frac{1}{2} \sigma^2 A^2 Be^{AW}.
\]
The \(Be^{AW}\) terms cancel, yielding the equation in \(A\) and \(B\),
\[
\rho = -(A/a) + (rW + (A/a)W + (1/a) \ln(AB/a))A + \frac{1}{2} \sigma^2 A^2.
\]
Equating coefficients of \(W\), we find
\[
0 = r + (A/a)
\]
and
\[
\rho = -(A/a) + (A/a) \ln(AB/a) + \frac{1}{2} \sigma^2 A^2.
\]
The first equation yields the solution \(A = -ra\). The second equation can then be solved for \(B\):
\[
\frac{\rho}{r} = 1 - \ln(-rB) + \frac{1}{2} \sigma^2 ra^2
\]
or
\[
B = -\frac{e^{\frac{\rho}{r} - \frac{1}{2} \sigma^2 ra^2}}{r}
\]
The optimal consumption process is
\[
c = rW - \frac{1 - \frac{\rho}{r} + \frac{1}{2} \sigma^2 ra^2}{a} \tag{4.2}
\]
Observe in passing that increasing the variance $\sigma^2$ or risk aversion $a$ (if $r < \rho$) decreases consumption and implicitly increases investment in the safe asset.

**Resource constraints and the equilibrium interest rate.** If the portfolio model is solved by an agent in a credit equilibrium the interest rate must clear the market. If there is a continuum of agents with independent endowment processes, the interest rate must result in zero drift for the consumption process. The zero-drift condition imposes a restriction that results in a solution for $r$.

The consumption process is the rent from wealth less a constant term. One can view this constant term as an insurance premium. The rent from wealth is a trend plus the accumulation of endowment. In an equilibrium with no aggregate dynamics as with this model, there cannot be drift for all individuals. Thus the *equilibrium* interest rate requires

$$\frac{1 - \frac{\rho}{r} + \frac{1}{2} \sigma^2 r a^2}{a} = 0$$

with solution

$$r = \frac{-1 \pm (1 + 2 \rho \sigma^2 a^2)^{1/2}}{\sigma^2 a^2}$$

The interest rate is lowered by risk aversion: the equation for the interest rate can be expressed as

$$r = \frac{\rho}{1 + \frac{1}{2} \sigma^2 a^2 r}$$

In the limiting case of zero risk aversion ($a = 0$), $r = \rho$, and otherwise $r < \rho$. This is similar to the finding in [14]; there, the principal’s discount factor was the same as the interest rate, and the interest rate was lower than the subjective rate of time discount, because the lower rate was needed to satisfy the incentive constraints. 3

**Transversality condition.** As the problem is set up it lacks an explicit liquidity constraint, so the optimal policy for the individual is to set $c = \infty$. Imposing the requirement that the long-run budget constraint be obeyed for each individual is expressed as follows:

$$E \left[ \int_t^\infty e^{-rs} (y(s) - c(s)) ds + e^{-rt} W(t) \right] = 0$$

for every $t$. That is, one cannot build up a stock of wealth at infinity which has a nonzero discounted value. If this condition is required to hold for each individual, then the credit equilibrium is feasible, as no outside resources are needed to sustain it. Optimality is satisfied with this requirement if a standard transversality condition is satisfied. The Lagrange

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3 The cost of information can be quantified as the ex ante welfare loss from the revelation constraint relative to the full-pooling contract that is possible under full information and the condition $\rho < a^2 \sigma^2 / 2$. For example, let $\sigma = .5$, $a = .75$, and $\rho = .06$. Then $r = .0597$, only slightly less than $\rho$, but the welfare cost as a fraction of ex-ante utility, $\frac{1 - \frac{1}{2} \sigma^2 r a^2}{\rho}$, is 7% of ex-ante discounted utility.
multiplier associated with (4.6), \( \hat{x} \), satisfies \( \hat{x} = V'(W) \), and therefore the transversality condition can be stated as

\[
\lim_{t \to \infty} E[e^{-\rho t} V'(W(t)) W(t)] = 0
\]  
(4.7)

Since \( V'(\cdot) \) is of exponential form it will dominate its argument. The transversality condition will therefore be satisfied if \( \lim_{t \to \infty} E[e^{-\rho t} V'(W(t))] = 0 \) holds. Filling in the structure of \( V'(W) \) under the zero-drift assumption yields

\[
\lim E e^{-\rho t} e^{-r \varepsilon(t)} = 0
\]
or

\[
\lim e^{(-\rho + \frac{1}{2} r^2 a^2 \sigma^2)t} = 0
\]

When \( r = 1 \) (no risk sharing possible due to private information) this is just the standard inequality \( \rho > a^2 \sigma^2/2 \) needed for convergence. When \( 0 < r < 1 \), the above condition is satisfied by the zero-drift interest rate, as demonstrated in the following proposition.

**Proposition 4.1:** If the parameters of the economy are such that the zero-drift interest rate exceeds zero, then the convergence condition \( \rho > \frac{a^2 \sigma^2}{2} \) is satisfied.

**Proof:** The zero-drift interest rate satisfies

\[
r = \frac{\rho}{1 + \frac{1}{2} a^2 \sigma^2 r}
\]

Substituting the requirement \( \rho > a^2 \sigma^2 r^2 / 2 \) results in the inequality

\[
r > \frac{a^2 \sigma^2 r^2 / 2}{1 + \frac{1}{2} a^2 \sigma^2 r}
\]
or

\[
r + \frac{1}{2} a^2 \sigma^2 r^2 > \frac{1}{2} a^2 \sigma^2 r^2
\]
or \( r > 0 \).

Borrowing without limit violates the transversality condition because the exponential term then grows to infinity. Imposing the transversality condition is therefore equivalent to imposing the dynamic budget constraint. Given that individuals would rather not obey this budget constraint and borrow without limit, how is this condition imposed? One interpretation is that there is an external institution that observes the optimal consumption and wealth policy functions chosen at time zero by the individuals, and then checks whether these functions satisfy the budget constraint in the asymptotic sense of (4.6), and requires subsequent adherence to those policies. Such an external institution must have the ability to observe each individual’s behavior to the appropriate degree and must also be robust to any desire on the individual’s part to subsequently defect from the implicit contract. The observability notion was already covered in the private-information contract; the requirement of nondefection will now be examined.
5. DEFLECTION

Consider an individual who participates in a credit equilibrium, and whose endowment has grown extremely large. The credit equilibrium, like its equivalent private-information contract, will in effect tax that individual, encouraging saving: that saving constitutes a transfer to individuals with low endowment, who borrow. As such, the high-endowment individual might be tempted to renege on the contract—to cease saving—and consume only his endowment. Since the endowment process here has high persistence, the benefit from this strategy would persist for a long time. However, that individual would also face the risk associated with consuming endowment, and this increased risk would have to be weighed against the gain from avoiding the tax.

From a contract perspective, the thought experiment just described constitutes defection from the contract, and would incur a punishment. Defection can be prevented if a sufficiently strong punishment can be devised that is internal to the payoff structure of the model. If the individuals in the economy are suitably patient, this can be done in a very conventional way. Any individual who defects will suffer a grim punishment: the permanent denial of insurance as it operates through the contract, forcing the defector to consume his endowment permanently. 4 When the condition \( \rho < a^2 \sigma^2 / 2 \) holds, the discounted value of consumption is \(-\infty\):

\[
E \left[ - \int_0^\infty e^{-\rho t} e^{-a \sigma z(t)} dt \right] = - \int_0^\infty e^{-\rho t} e^{\frac{1}{2} a^2 \sigma^2 t} dt = - \infty
\]

The credit equilibrium results in a finite utility payoff. Without loss of generality consider an individual with zero initial wealth. The individual will then have two components of consumption: \((1 - r)y(0)\), and \(ry(t)\). Recalling that \(y(t) = \sigma z(t)\), utility is then

\[
-e^{-a(1-r)y(0)} E \left[ - \int_0^\infty e^{-\rho t} e^{-a \sigma z(t)} dt \right] \sim - \int_0^\infty e^{-\rho t} e^{\frac{1}{2} a^2 \sigma^2 t} dt = - \frac{1}{\rho - \frac{1}{2} a^2 \sigma^2}
\]

and by Proposition 4.1 this is finite. The threat of an infinite loss of utility will therefore deter defection. This reasoning demonstrates the following proposition:

PROPOSITION 5.1: Let \( \rho < \frac{1}{2} a^2 \sigma^2 \). Then a credit equilibrium in which the interest rate satisfies the zero-drift condition and in which consumption is \((1 - r)y(0) + ry(t)\) is a subgame perfect Nash equilibrium.

This is a folk theorem result. Note that the equilibrium fails to provide full insurance as would be the case if the interest rate were zero; this reflects the operation of the revelation constraint in the corresponding private-information contract.

If individuals are impatient, that is if the inequality \( \rho > \frac{1}{2} a^2 \sigma^2 \) holds, the lost-insurance cost of defecting becomes finite. With only a finite punishment, it is no longer

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4 This equilibrium notion was first used in a heterogeneous-agent dynamic insurance model in [16]. Other models such as [7] or [8] or [1] solve a planner problem in the spirit of Stokey and Lucas [9], in which one individual optimizes while satisfying utility continuation constraints associated with the other agents. Nondefection is an explicit constraint in which utility must not be dominated by autarkic utility. Ligon, Thomas and Worrall [8] find that storage can actually reduce welfare because when storage is accumulated the temptation of individuals with high levels of storage to defect increases—their defection constraint binds.
directly obvious that a Nash equilibrium is sustainable. The analysis of this more complex situation will now be taken up.

**Defection when \( \rho \) is large.** If \( \rho > \frac{1}{2}a^2\sigma^2 \), the analysis of defection must account for defection occurring at some finite endowment. For this reason, a smooth pasting approach will be used. The first step in the analysis will be to note that the defection incentive is indeed positive for the basic credit equilibrium model that has been outlined. A modification of the contract will then be described that preserves a risk-sharing arrangement.

At this point, in order to conserve notation, it will be assumed that \( \sigma = 1 \). Beginning the smooth pasting construction, the differential equation for the portfolio value process is

\[
V(z) = B_1 e^{\sqrt{2\rho z}} + B_2 e^{-\sqrt{2\rho z}} - \frac{1}{\rho - \frac{r^2 a^2}{2}} e^{-a z}
\]

(5.1)

This holds because the endowment process is \( z(t) \), but this is also the form of the wealth process \( W(t) \), and because in a market-clearing equilibrium, \( c = rW \).

The solution above presumes zero initial wealth without loss of generality, because of the absence of wealth effects for exponential utility.

If there is defection then intuitively it must occur when endowment is high. When endowment is low, the portfolio smooths out the loss—if \( r < 1 \)—and this will deter defection. It is therefore reasonable to suppose that \( B_2 = 0 \). As \( z \to -\infty \), the value then becomes dominated by the utility term rather than the \( B_1 \) boundary term.

If there is a defection boundary \( z^* \), then it will be recoverable from the smooth pasting boundary conditions. There are two conditions: a level condition and a derivative condition, and they jointly determine \( B_1 \) and \( z^* \):

\[
B_1 e^{\sqrt{2\rho z^*}} - \frac{1}{\rho - \frac{r^2 a^2}{2}} e^{-a z^*} = -\frac{1}{\rho - \frac{a^2}{2}} e^{-a z^*}
\]

(5.2)

\[
B_1 \sqrt{2\rho} e^{\sqrt{2\rho z^*}} + \frac{a r}{\rho - \frac{r^2 a^2}{2}} e^{-a r z^*} = \frac{a}{\rho - \frac{a^2}{2}} e^{-a z^*}
\]

(5.3)

Subtracting these two conditions eliminates \( B_1 \) and leaves an equation in \( z^* \):

\[
-\frac{1}{\rho - \frac{r^2 a^2}{2}} e^{-a r z^*} (1 + \frac{a r}{\sqrt{2\rho}}) = -\frac{1}{\rho - \frac{a^2}{2}} e^{-a z^*} (1 + \frac{a}{\sqrt{2\rho}})
\]

(5.4)

The left hand side is just a carbon copy of the right hand side with \( ra \) substituted for \( a \).

If the functions in (5.4) are non-monotone then it is possible for a nontrivial solution to exist. Equation (5.4) can be simplified:

\[
e^{-a(r-1)z^*} = \frac{\rho - \frac{r^2 a^2}{2} 1 + \frac{a}{\sqrt{2\rho}}}{\rho - \frac{a^2}{2} 1 + \frac{a r}{\sqrt{2\rho}}}
\]

(5.4)'

Taking logs,

\[
(1 - r) a z^* = \ln \left( \frac{\rho - \frac{r^2 a^2}{2} 1 + \frac{a}{\sqrt{2\rho}}}{\rho - \frac{a^2}{2} 1 + \frac{a r}{\sqrt{2\rho}}} \right).
\]

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This simplifies further upon noting that
\[
\rho - \frac{a^2}{2} = \rho \left(1 - \frac{a}{\sqrt{2\rho}}\right)\left(1 + \frac{a}{\sqrt{2\rho}}\right)
\]
\[
(1 - r)az^* = \ln \left(1 - \frac{ra}{\sqrt{2\rho}}\right)
\]
The argument on the right hand side exceeds 1 for \(r < 1\), and so there is a well-defined finite \(z^*\) that solves the equation. However a necessary condition is that
\[
\frac{1 - \frac{ra}{\sqrt{2\rho}}}{1 - \frac{a}{\sqrt{2\rho}}}
\]
is positive. If \(r < 1\) in equilibrium, then this requirement is satisfied by the familiar condition \(\rho > a^2/2\). For example, for the parameter values \(\rho = .09\) and \(a = .05\) the equilibrium interest rate is \(0.09989\) and \(z^* = 2.5203\).

These values are a decreasing function of \(\rho\)—that is, decreased patience results in a smaller threshold of defection \(z^*\) and a decreased duration in the risk-sharing contract provided by an asset equilibrium; this is a folk-theorem result. Indeed, for large values of \(\rho\), the defection threshold becomes negative and there is no range in which assets will be held.

**Unsustainability of equilibrium.** It is now evident that, when there is insufficient patience, the credit equilibrium as posited cannot be sustained. That is because defection occurs when endowment is high, but not symmetrically when it is low. Therefore the resource constraint that averages high and low endowments so as to ensure zero drift, as expressed in (4.1-5), fails. Is there an interest rate that generates a resource balance but allows defection? The following section demonstrates that with additional structure this is the case. The additional structure weakens the insurance, but retains enough insurance to cause potential defectors to fear its loss should they defect.

### 6. A CREDIT-SEGMENT HYBRID

In the credit model, the equilibrium is such that consumption is a fraction of endowment. Here I extend the idea to incorporate segments, the construction in [4]. Intuitively, a segment construction consists of segments \(\{S_k\}_{k=\infty}^\infty\), with \(S_k = [y_{k-1}, y_{k+1})\) and \(y_k = k\Delta\), where \(\Delta\) is a fixed constant. When an individual’s endowment process hits the center \(y_k\) of the segment \(S_k\), he commences sharing his endowment with individuals whose endowments previously have hit \(y_k\) and have yet to hit one of the boundaries of the segment, \(y_{k-1}\) and \(y_{k+1}\). Until he hits these boundaries, which occurs at \(T = T_{y_{k-1}} \land T_{y_{k+1}}\), the agent continues to share in this manner. When he hits the boundary, say at \(y_{k+1}\), he commences sharing in the new segment \(S_{k+1}\).

In [4], it was demonstrated that with full information, and full pooling of endowment within each segment, defection could be deterred. Under private information, full pooling
is not incentive compatible. The ability of the partial pooling within segments that is feasible under private information to deter defection will now be considered.

Let the credit contract extend to \( T = T_{y_{k-1}} \land T_{y_{k+1}} \) instead of infinity, with initial endowment \( y_k \). Then the credit contract will provide consumption \( r(y(t) - y_k) \) net of the segment consumption \( y_k \). Consumption can be expressed in two ways:

\[
c(t) = y_k + r(y(t) - y_k) = (1 - r)y_k + ry(t)
\]

In the first expression consumption consists of the segment consumption plus an uninsured residual; in the second expression consumption consists of a weighted average of segment consumption, which is insured, and a fraction of current endowment, which is uninsured. The second expression captures the idea that the credit component of consumption does not jump. It is clear that individuals will in fact choose credit in the same fashion as they do in the pure credit equilibrium, despite the finite stopping time horizon \( T \).

\textbf{Defection.} Defection in this sort of contract means that the segment part of consumption will revert to endowment, so consumption consists of \((1 - r)y(t) = y(t)\). Can defection be deterred? It has already been established in [4] that grim punishments work to deter defection of the segment contract equilibrium if there are infinitely many segments. It is also known that this punishment is excessive, in that the same outcome can be accomplished with finite punishments. Can the excess punishment power of the grim punishment be applied to the lending part of the equilibrium? It seems likely; if this is so, then the segment model really is also a model of a modified credit equilibrium in which credit and a kind of localized risk sharing coexist. Clearly the insurance here is not as good as a pure segment equilibrium—but it is known from the previous sections that a credit equilibrium mimics a contract in which information is private and must be revealed. This motivates the reduced insurance: it is the price of revelation.

The value of contractual consumption is

\[
v(y|y \in S_k) = -E \left[ \int_0^{T_{y_{k-1}} \land T_{y_{k+1}}} e^{-\rho s} e^{-a((1-r)y_{k} + r y(s))} ds + e^{-\rho T_{y_{k-1}} \land T_{y_{k+1}}} v(y(T_{y_{k-1}} \land T_{y_{k+1}}))|y(0) = y_k \right]
\]

It will be helpful to calculate the following quantity:

\[
\zeta(y) = E \left[ \int_0^{T_{y_{k-1}} \land T_{y_{k+1}}} e^{-\rho s} e^{-a y(s)} ds | y \right]
\]

which has general solution

\[
\zeta(y) = -\frac{e^{-a y}}{\rho - (a y)^2} + B_1 e^{\sqrt{2} y} + B_2 e^{-\sqrt{2} y}
\]

The recursion can be restated as follows:

\[
v(y)
\]
\[ E \left[ -e^{-\alpha y} \int_0^{T_{y_{k-1}} \wedge T_{y_{k+1}}} e^{-\rho s} e^{-\alpha(y(s) - y)} ds + e^{-\rho T_{y_{k-1}} \wedge T_{y_{k+1}}} v(y(T_{y_{k-1}} \wedge T_{y_{k+1}})) \big| y(0) = y_k \right] \]

\[ = -e^{-\alpha y} \zeta(y) + E \left[ e^{-\rho T_{y_{k-1}} \wedge T_{y_{k+1}}} v(y(T_{y_{k-1}} \wedge T_{y_{k+1}})) \big| y(0) = y_k \right] \]

At the midpoint of the segment, define \( v(y = y_k) = v_k \), and the recursion becomes

\[ v_k = -e^{-\alpha y_k} \zeta(y_k) + E \left[ e^{-\rho T_{y_{k-1}} \wedge T_{y_{k+1}}} v(y(T_{y_{k-1}} \wedge T_{y_{k+1}})) \big| y(0) = y_k \right] \]

The expected value of the integral is independent of \( y \), so for \( y = y_k \) it is possible to normalize:

\[ \bar{v}_k = -e^{-\alpha y_k} + E \left[ e^{-\rho T_{y_{k-1}} \wedge T_{y_{k+1}}} \bar{v}(y(T_{y_{k-1}} \wedge T_{y_{k+1}})) \big| y(0) = y_k \right] \quad (6.1) \]

This has a structure similar to the corresponding model of [4]. Before proceeding, it is necessary to complete the analysis of \( \zeta(y) \).

**Solving for \( \zeta(y) \).** The general solution of \( \zeta(y) \) is

\[ \zeta(y) = -\frac{e^{-\alpha y}}{\rho - \frac{(\alpha r)^2}{2}} + B_1 e^{\sqrt{2\rho y}} + B_2 e^{-\sqrt{2\rho y}} \]

The boundary conditions for a fixed segment of size \( 2D \) are

\[ \zeta(D) = \zeta(-D) = 0 \]

or

\[ \frac{e^{-\alpha D}}{\rho - \frac{(\alpha r)^2}{2}} + B_1 e^{\sqrt{2\rho D}} + B_2 e^{-\sqrt{2\rho D}} = 0 \]

\[ \frac{e^{\alpha D}}{\rho - \frac{(\alpha r)^2}{2}} + B_1 e^{-\sqrt{2\rho D}} + B_2 e^{\sqrt{2\rho D}} = 0 \]

These equations can be solved for the \( B_i \):

\[
\begin{pmatrix}
  e^{\sqrt{2\rho D}} & e^{-\sqrt{2\rho D}} \\
  e^{-\sqrt{2\rho D}} & e^{\sqrt{2\rho D}}
\end{pmatrix}
\begin{pmatrix}
  B_1 \\
  B_2
\end{pmatrix}
= \begin{pmatrix}
  \frac{e^{-\alpha D}}{\rho - \frac{(\alpha r)^2}{2}} \\
  \frac{e^{\alpha D}}{\rho - \frac{(\alpha r)^2}{2}}
\end{pmatrix}
\]

Defining

\[ \lambda \equiv e^{-\sqrt{2\rho D}} \]

the equation becomes

\[
\begin{pmatrix}
  B_1 \\
  B_2
\end{pmatrix}
= \frac{1}{\rho - \frac{(\alpha r)^2}{2}} e^{\sqrt{2\rho D}} \begin{pmatrix}
  \lambda \\
  -\lambda
\end{pmatrix}
= \begin{pmatrix}
  e^{\sqrt{2\rho D}} & -e^{-\sqrt{2\rho D}} \\
  -e^{\sqrt{2\rho D}} & e^{-\sqrt{2\rho D}}
\end{pmatrix}
\begin{pmatrix}
  e^{-\alpha D} \\
  e^{\alpha D}
\end{pmatrix}
\]

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\[
= - \frac{1}{\rho - \frac{(a r)^2}{2} \lambda^2} \left( \lambda^{-1} - \lambda \right) \left( e^{-a r D} \right) \\
= - \frac{1}{\rho - \frac{(a r)^2}{2} \lambda^2} \left( \lambda^{-1} e^{-a r D} - \lambda e^{a r D} \right) \\
\]

The full solution for \( \zeta(y) \) is therefore

\[
\zeta(y) = - \frac{1}{\rho - \frac{(a r)^2}{2} \lambda^2} \left( e^{-a r y} - (\lambda^{-1} e^{-a r D} - \lambda e^{a r D}) e^{\sqrt{2} p y} - (\lambda e^{-a r D} - \lambda^{-1} e^{a r D}) e^{-\sqrt{2} p y} \right) \frac{1}{\lambda^2 - \lambda} \\
\]

This formula can be fed into the formula for \( v(y) \). Observe that at the extreme of \( y = D \) or \(-D\) the formula reduces to zero.

Solving for \( v_k \). The equation for \( \tilde{v}_k \), (6.1) has almost exactly the same structure as the corresponding model of [4]. Now write the normalized difference equation:

\[
\tilde{v}_k = -e^{-a y_k} + r(\rho, D, D) \tilde{v}_{k-1} + r(\rho, D, D) \tilde{v}_{k+1} \\
\]

This is not identical to the corresponding equation in [4] because the \((1 - q)\) factor is subsumed into the integration in \( \zeta(0) \). Since the weighting of the current payoff occurs in the \( \zeta(0) \) term, the correct form is thus

\[
v_k = -\zeta(0) e^{-a D k} \frac{1 + \lambda^2}{\lambda^2} \frac{1}{1 - \lambda^2} \left( 1 - \frac{\lambda^2}{(1 - \lambda e^{a D})(1 - \lambda e^{-a D})} \right) \\
\]

As verification for the result, the equivalence of the \( r = 1 \) case with pure autarky and the \( r = 0 \) case with the pure segment contract model will now be established.

\( r = 0 \). At a value of \( r = 0 \) (pure segment contract) it is straightforward to demonstrate

\[
B_1 + B_2 = \frac{1}{\rho} \frac{2}{e^{2 \sqrt{2} p D} + e^{-2 \sqrt{2} p D}} \frac{1}{\rho} \frac{1}{\cosh(2 \sqrt{2} p D)} = \frac{1}{\rho} \frac{2}{\lambda + \lambda^{-1}} = \frac{1}{\rho} q \\
\]

This can be demonstrated to relate directly to \( 1 - q \), as stated. Thus:

\[
\zeta(0|r = 0) = -\frac{1}{\rho} (1 - q) \\
\]

so

\[
v_k = -\frac{e^{-a D k}}{\rho} \frac{(1 - \lambda)^2}{(1 - \lambda e^{a D})(1 - \lambda e^{-a D})} \\
\]

exactly as in [4].
\[ r = 1. \] If \( r = 1 \) (autarky) then the solution for \( B_1 \) and \( B_2 \) is
\[
\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \frac{1}{c^2 \sqrt{2\rho D} - e^{-2\sqrt{2\rho D}}} \begin{pmatrix} e^{\sqrt{2\rho D}} & -e^{-\sqrt{2\rho D}} \\ -e^{-\sqrt{2\rho D}} & e^{\sqrt{2\rho D}} \end{pmatrix} \begin{pmatrix} e^{-aD} \\ e^{aD} \end{pmatrix}
\]
Recalling that \( \lambda \equiv e^{-\sqrt{2\rho D}} \), this can be expressed as follows:
\[
\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \frac{1}{\rho - \frac{a^2}{2}} \frac{1}{\lambda^2 - \lambda^{-2}} \begin{pmatrix} \lambda^{-1} & -\lambda \\ -\lambda & \lambda^{-1} \end{pmatrix} \begin{pmatrix} e^{-aD} \\ e^{aD} \end{pmatrix}
\]
with solution
\[
\frac{1}{\rho - \frac{a^2}{2}} \frac{1}{\lambda^2 - \lambda^{-2}} \begin{pmatrix} \lambda^{-1} e^{-aD} - \lambda e^{aD} \\ -\lambda e^{-aD} + \lambda^{-1} e^{aD} \end{pmatrix}
\]
At \( v_k \), the sum \( B_1 + B_2 \) is
\[
B_1 + B_2 = \frac{e^{aD} + e^{-aD}}{\lambda^{-1} + \lambda}
\]
and therefore
\[
\zeta(0) = -\frac{1}{\rho - \frac{a^2}{2}} \left( 1 - \frac{e^{aD} + e^{-aD}}{\lambda^{-1} + \lambda} \right)
\]
and
\[
v_k = \zeta(0) \frac{1 + \lambda^2}{1 - \lambda^2} \left( \frac{\lambda e^{aD}}{1 - \lambda e^{aD}} + \frac{1}{1 - \lambda e^{-aD}} \right) e^{-aDk}
\]
\[
= -\frac{e^{-aDk}}{\rho - \frac{a^2}{2}}
\]
which is exactly the autarky value.

The conclusion so far is that the credit-segment hybrid is a convex combination of the autarky solution and the pure segment solution. The nonlinearity of the \( \zeta(0) \) term for \( 0 < r < 1 \) however means that it is not a linear convex combination. In particular it must be verified that for the equilibrium \( r \)—the zero-drift solution—the insurance value deters defection. Since the insurance value is reduced relative to the pure segment contract, knowing that a grim punishment deters defection in the full-segment model is not sufficient: it must be demonstrated for the equilibrium-\( r \) case per se. Only then can it be concluded that defection with a finite span of punishment is possible.

**Defection value within a segment.** From an arbitrary point within a segment, the defection value can be checked. Rather than using the smooth pasting approach, I compare the value of defection with the value of never defecting, in order to examine the effect of \( r > 0 \). The value of defection is the value of autarky already examined:
\[
v_A(y) = -\frac{e^{-ay}}{\rho - \frac{a^2}{2}}
\]
The contract value at \( y \) when \( y \) is interior to the segment is

\[
v_C(y) = -e^{-\alpha y} \zeta(y) + r(\rho, y - y_{k-1}, y_k - y)v_{k-1} + r(\rho, y_{k+1} - y, y - y_k)v_{k+1}
\]

(6.2)

where \( v_C \) is the contract value.

**Equilibrium \( r \).** Because the uninsured part of endowment is \( ry(t) \) in the credit-segment hybrid, the insurance value of the segments shrinks as \( r \) increases to 1.0, and the defection value also shrinks. Moreover, since the defection value is strictly negative and finite for \( r < 1 \) there is enough room for a finite span of punishment. The construction in [4] is therefore sufficient to support the private information contract and its equivalent credit equilibrium for any parameters such that \( r < 1 \) in equilibrium.

The following result holds.

**Proposition 6.1:** There exists a segment equilibrium that is free from defection and that supports the credit equilibrium at the feasible interest rate.

**Proof:** Equation (6.2) has five components. The two hitting probabilities, \( r_{y,y_{k-1}} \) and \( r_{y,y_{k+1}} \) are unaffected by the interest rate \( r \). The continuation values \( v_{k-1} \) and \( v_{k+1} \) are each of the form \( \zeta(0)\hat{v}_{k+1} \), and \( \hat{v} \) is independent of \( r \). Finally, the first term is has the factor \( \zeta(y) \). It suffices to demonstrate that \( \zeta(0) \) and \( \zeta(y) \) are monotone in \( r \). But this is true since

\[
\frac{\partial}{\partial r} \zeta(y) = -E \left[ \int_0^{T_{y_{k-1}} \wedge T_{y_{k+1}}} e^{-\rho s} e^{-\sigma y(s)} a_y(s) ds | y \right] > 0
\]

This follows from the symmetry of the Gaussian density as a function of time, and the asymmetry of the exponential. Therefore as \( r \) is decreased from the autarky value of \( r = 1, \zeta(y) \) decreases. This is true for \( \zeta(0) \) as well. Therefore the right hand side of (6.2) is monotonically decreasing in \( r \). By Proposition 6.1 of [4], a grim punishment is sufficient to deter defection; since the contract value is decreasing in \( r \), the same result holds here, and \( v_C(y) > v_A(y) \) for \( r \) in a credit equilibrium. ■

It is evident from additional results in [4] that less-than-grim punishments can also feasibly sustain the contract.

### 7. Discussion

In the presence of private information the value of insurance is reduced, and this reduction expresses itself as a credit component of an equilibrium contract. It is standard to think of credit as the expression of insurance—but here the credit component of consumption, \( ry(t) \), is the uninsured part of endowment! The insured part is the segment contract, which is a less familiar construction. What is surprising is that both components are needed. In real economies the theory of assets needs to account for the background institutions that support the asset markets, and also to account for the informational constraints that cause credit-like assets to exist in the first place.
The credit-segment hybrid model can be regarded as having two types of asset. The first is the credit asset, which behaves like bonds, and the second is the segment redistribution, which can be regarded as equity. The segments act like equity in the sense that all realizations are accounted for in the Arrow-Debreu sense, with (temporary) perfect risk sharing.

Both types of asset are necessary: the credit asset is needed because of the private information constraint, and the equity asset is needed because of the deflection constraint. Because both are needed, the Modigliani-Miller [11] theorem, which states that they are equivalent, is violated.

Because the Modigliani-Miller theorem is violated, the model makes possible a comparison of the differential rates of return on bonds and equity. The interest rate for credit is given by a formula, with \( r < \rho \); the interest rate on equity is just the discount factor \( \rho \); the segment component of consumption is constant inside the segment. The model therefore generates an equity premium. It is very different in spirit from the standard equity premium as in [10] however: the “risk-free-rate” \( r \), the interest rate on credit and bonds, is what is influenced by risk \( (\sigma^2) \) and also risk aversion \( (a) \); the equity rate is not. \(^5\)

The standard model has an equity premium because of the nondiversifiable risk in the model. However, the model should then have risk-free bonds in zero net supply; no such bonds could survive in an equilibrium. In the present model bonds are in fact traded to reflect the evolution of the heterogeneous endowments. Since both bonds and equity are present in a well-defined equilibrium sense, the equity premium also is clearly defined.

The equity premium remains a “puzzle” because the standard model is constrained to add only aggregate risk, and that aggregate risk is strictly defined by the empirical value of aggregate output fluctuations; although there have been occasional extreme values due to war and recession, this value is relatively low. The equity premium in the present model is determined by the variance of idiosyncratic endowment—aggregate fluctuation is zero. The empirical magnitude of idiosyncratic risk is much higher than that of aggregate risk, and so this constraint on explaining the equity premium is relaxed in the present model.

The contract model was constructed in order to elucidate how redistribution of endowment could occur, using its equivalence with insurance and risk-sharing, under the dual constraints of private information and lack of commitment. A kind of welfare theorem has emerged: a credit equilibrium replicates the optimal private-information contract, and the disparity of consumption increases over time within this equilibrium. This reflects the operation of the private-information constraint and the consequent sacrifice of some of the insurance possibilities. Furthermore, requiring that individuals remain permanently and voluntarily committed to the contract requires a further weakening of the insurance possibilities due to segment formation, if there is insufficient patience. Because of the resemblance of the credit equilibrium to real credit equilibria, including the temporarily increasing wealth dispersion, the model is potentially testable. Should such tests succeed, empirical measures of the cost of extracting private information and of sustaining commitment can be obtained.

\(^5\) The same finding holds in a constant relative risk aversion model in which endowment is a geometric Brownian motion. If \( a \) is the coefficient of constant relative risk aversion, the interest rate is \( \rho + a(1-a)\sigma^2/2 \). For values of \( a \) in excess of 1.0 the bond rate \( r \) is less than the equity rate \( \rho \).
REFERENCES


