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Abstract

Consider an exchange economy with n traders and k goods. Suppose that there exists C^2 mappings $p(\cdot)$ and $x(\cdot)$ that specify for each initial endowment w of the k goods to the n traders a price vector $p(w)$ and a vector of net trades $x(w)$ that define a competitive equilibrium. Three systems of determinant equations are derived that the derivatives of $p(\cdot)$ and $x(\cdot)$ necessarily satisfy regardless of the underlying preferences that determine these mappings. Each of the three systems of equations fundamentally reflects the interaction among traders in the competitive mechanism.

1 Introduction

Consider an exchange economy with n traders and k goods. Let w_i denote the endowment of trader i and let $w = (w_i)_{1 \leq i \leq n}$ denote an endowment for the economy. Assume that w lies in some open subset W of \mathbb{R}_{++}^{kn} . For fixed preferences of the traders, suppose that there exists C^2 mappings $p(\cdot)$ and $x(\cdot)$ that specify for each w in W a price vector $p(w)$ and a vector of net trades $x(w)$ for the n traders that together define a competitive equilibrium. I derive in this paper three systems of determinant equations that the derivatives of $p(\cdot)$ and

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$x(\cdot)$ necessarily satisfy regardless of the underlying preferences that determine these mappings.

The three systems are based upon the following three principles:

1. a trader's equilibrium net trade depends upon the endowments of the other traders only insofar as they affect $p(\cdot)$;
2. $p(\cdot)$ is invariant and $x(\cdot)$ changes in a particular way as w is perturbed in a manner consistent with each trader's budget equation and the balance of net trades;
3. $p(\cdot)$ is realized by the competitive mechanism, which has distinctive mathematical properties by virtue of being a decentralized mechanism.

Each of these three principles is developed more fully below as part of deriving the system in question. It can be noted at this point that each of the systems fundamentally reflects interaction among traders in the competitive mechanism and therefore cannot be derived solely by considering a single trader's behavior.

Equations are thus derived here concerning competitive market aggregates (prices and net trades) as functions of physical quantities (endowments). The paper contributes to the long-standing research program whose objective is to derive testable implications of general equilibrium theory. Most of the research in this area has been negative vis-à-vis this objective in the sense that market aggregates and individual trader behavior have typically been shown to satisfy only the most rudimentary of restrictions once the preferences of the traders are varied. Examples include the Debreu-Mantel-Sonnenschein Theorem,¹ which shows that the excess demand function in an exchange economy is restricted only near the boundary of the price simplex, and the related work of Mas-Colell (1977), who showed that any compact and nonempty subset of the interior of the price simplex is the equilibrium price set of some economy. This work takes the number of traders, their preferences, and their endowments as variables that are selected to support a given mapping or set as a market aggregate. The present paper takes the preferences of n traders as fixed in their determination of the mappings $p(\cdot)$ and $x(\cdot)$. Explicit determinant equations on the first and second derivatives of $p(\cdot)$ and $x(\cdot)$ are then derived that are independent of the traders' preferences, i.e., the equations hold for all $p(\cdot)$ and $x(\cdot)$ obtained from traders' preferences, regardless of what those preferences may be. The preferences of the traders are effectively eliminated in deriving the equations, with the end result being a rather striking set of restrictions on $p(\cdot)$ and $x(\cdot)$.

The research program cited above has recently been revived by Brown and Matzkin (1996), who proved the existence of a set of polynomial equalities and inequalities that m points selected from the endowment - competitive price set

¹The relevant references are Debreu (1974), Mantel (1974), and Sonnenschein (1972, 1973).

necessarily satisfy.^{2,3} Besides being the source of my interest in this research program, a connection between this paper and Brown and Matzkin (1996) is that the present paper also concerns the endowment - competitive price set for economies in which this set is locally the graph of a C^2 mapping between endowments and the competitive price vector. The equations derived on the first and second derivatives of this mapping can be interpreted as restricting the structure of the competitive price - endowment set, which is the objective of the Brown-Matzkin paper. The two papers are distinguished in (i) my use of the differential approach, (ii) their investigation of the econometrics of testing their result, and (iii) my inclusion of the equilibrium net trade mapping $x(\cdot)$ as part of the analysis. With regards to point (iii), $x(w)$ is addressed here in part because it is a mapping from one observable variable to another, and hence it may be empirically meaningful. It is also the case that $x(w)$ is determined by $p(w)$ and the traders' excess demand mappings. The excess demand mappings are critical in this paper because they are the fulcrum on which the study of decentralization in the competitive mechanism rests. Addressing $x(w)$ thus follows naturally from addressing the constraints on $p(w)$ that follow from its realization by the competitive mechanism.

Chiappori et al. (2002) is most closely related to the present work in methodology and objectives. Their paper adopts the local differential approach that is used herein. In their Theorem 1, which contains the results most directly relevant to this paper, they show: (i) the price mapping $p(w)$ determines each trader's marginal propensities to consume; (ii) these individual income effects satisfy a specific family of equations; (iii) $p(w)$ in fact formally determines the net trade mapping $x(w)$ and the preferences of the traders as long as $x(w)$ is generic in a specific sense.⁴ Though the broad aims and the methodology of my paper and theirs is the same, there is no overlap in the results of the two papers. In defense of my contribution, while $p(w)$ may formally define $x(w)$, it is certainly worthwhile to explicitly state the relationships that exist directly between these two mappings. This is addressed rather simply in the systems (5) and (17) below. Also, the Chiappori et al. approach does not directly constraint the mapping $p(w)$ itself. This is the main accomplishment of my paper and it is presented in the systems (16) and (27) below. In short, none of the systems of equations that are derived herein appear in Chiappori et al. (2002).

As mentioned in principle 3. above, the third system is derived by consider-

²The empirical meaningfulness of this research program may be questioned on the grounds that the endowments of traders are typically not collected in field market data. This is due partly to misrepresentation on the part of the traders but also because the set of traders who participate in a market may not be observable. Empirical testing of theoretical results concerning the endowment - competitive equilibrium set may thus be most feasible in experiments, where the set of participants and their endowments can be controlled.

³The papers of Nachbar (2001a, b) are recent contributions to this research program.

⁴Chiappori et al. (2002, Thm. 2) also show that prices as functions of the aggregate endowment are essentially indeterminate in the sense that an economy can be constructed that has a given mapping as its equilibrium price mapping. Since the goal of this line of research is to restrict prices as functions of endowment, their result demonstrates the necessity of information concerning individual endowments.

ing the realization of the price and net trade mapping through the competitive mechanism, as modeled initially in the work of Leonid Hurwicz and Stanley Reiter. The mathematics of the differential approach to mechanisms that is employed in this paper originated in Hurwicz, Reiter and Saari (1978) with subsequent development in Saari (1984). Most analysis of mechanisms focuses upon the privacy of preference information for a trader and its affect upon what can be accomplished by the group. Here, however, a trader's endowment constitutes his information while his preferences remain fixed. The privacy of his information and the consequent problem of incentives is not addressed in this paper. Instead, the third system stems from the observation that each trader can verify that a proposed trade is optimal for him given the price vector and knowing only his own endowment and not those of the other traders. This reflects the decentralization of decision-making in the competitive mechanism.

The paper is organized as follows. The model is addressed in the next section. The three systems of equations are then derived in sections 3-5. Section 6 illustrates these systems in an example with $k = 2$ goods and $n = 2$ traders, each of whom has a Cobb-Douglas utility function. This form of utility allows $p(\cdot)$ and $x(\cdot)$ to be calculated explicitly. Directions for future work are then discussed in the Conclusion.

2 The Model

Let $w_{i,t}$ denote trader i 's endowment of good t . Trader i 's endowment $w_i = (w_{i,t})_{1 \leq t \leq k}$ is an element of an open subset $W_i \subset \mathbb{R}_{++}^k$. Let $W = \prod_{i=1}^n W_i$ denote the set of possible endowments for the economy. The price vector p is normalized to lie in the $(k - 1)$ -dimensional price simplex $\Delta \subset \mathbb{R}^k$.

The preferences of the traders are fixed. They are assumed to be restricted sufficiently so that the following conditions hold.

1. There exists an open set $O \subset \text{Int } \Delta$ and a C^2 mapping $p : W \rightarrow O$ such that $p(w)$ is the unique competitive equilibrium price vector given the endowment w that lies in O .
2. Each trader i 's excess demand $\chi_i : O \times W_i \rightarrow \mathbb{R}^k$ is a C^2 mapping on $O \times W_i$.

These are the only restrictions on the economy in this paper. It is routine to place regularity conditions on utility functions of the traders so that assumptions 1. and 2. hold near a given $w^* \in \mathbb{R}^{nk}$ by virtue of the Implicit Function Theorem. I therefore do not pursue this topic in this paper.

Assumptions 1. and 2. hold throughout the paper. Two points should be noted concerning these assumptions. First, $p(w) \in \text{Int } \Delta$ insures that the equilibrium price of each good is positive in this paper. Second, the freedom to restrict prices by properly choosing W and $O \subset \text{Int } \Delta$ means that uniqueness of the competitive equilibrium for each choice of the endowment is *not* assumed

in assumption 1.. Rather, it asserts local uniqueness of the competitive price vector for $w \in W$, i.e., the equilibrium price vector $p(w)$ is isolated.

Let $\mathcal{E}(w) = (p(w), x(w))$ denote the competitive equilibrium mapping where $x : W \rightarrow \mathbb{R}^{nk}$ is the equilibrium net trade mapping corresponding to $p(w)$. Represent trader i 's net trade in $x(w)$ as $x_i(w) \in \mathbb{R}^k$ and let $x_{i,t}(w)$ denote his net trade in good t . The mapping $x(\cdot)$ is defined by

$$x_i(w) = \chi_i(p(w), w_i) \quad (1)$$

holding for each trader i . It is C^2 by virtue of 1. and 2. above. Because $p(w)$ and $x(w)$ define a competitive equilibrium, they necessarily satisfy the *balance* equation

$$\sum_{i=1}^n x_i(w) = 0 \quad (2)$$

and the *budget* equation

$$p(w) \cdot x_i(w) = 0 \quad (3)$$

for each trader i . Together with the normalization of prices to Δ , these familiar equations are of interest partly for the purpose of reducing the systems that are derived in this paper to verify that each restricts $p(\cdot)$ and $x(\cdot)$ in ways that these familiar equations on competitive equilibrium do not.

Let $p = (p_t)_{1 \leq t \leq k}$ so that p_t is the price of good t . Define $p_{-k}(w)$ as the mapping

$$p_{-k}(w) = (p_1(w), \dots, p_{k-1}(w))$$

consisting of all but the k th price. Because $p(w) \in \Delta$, the rows of the $k \times k$ matrix $D_{w_i} p$ sum to zero and hence this matrix has rank at most $k - 1$. On occasion, it will be assumed that $p(w)$ satisfies the regularity condition

$$\text{rank } D_{w_i} p_{-k}(w) = \text{rank } D_{w_i} p(w) = k - 1 \quad (4)$$

for each trader i .

Finally, each system of equations that is derived in this paper is summarized as a set of rank conditions on matrices. Each rank condition asserts that some $a \times b$ matrix whose entries are defined by $p(\cdot)$ and $x(\cdot)$ and for which $a \leq b$ has rank less than a . It is straightforward to show that such a rank condition is equivalent to $b - a + 1$ independent determinant equations on the matrix.⁵ Because representing a rank condition with an equivalent set of determinant equalities is routine, the systems are left in the summary form of rank conditions below.

⁵Let A be the $a \times b$ matrix in question. The condition $\text{rank } A < a$ can be represented as a system of independent equations by first selecting an $a \times (a - 1)$ submatrix \tilde{A} of A of maximal rank among all submatrices of A of this size. Successively place each of the $b - (a - 1) = b - a + 1$ remaining columns next to \tilde{A} and then take the determinant. Each of these $b - a + 1$ determinants equaling zero is necessary and sufficient for the matrix A to have rank less than a .

3 Invariance of $x_q(w)$ to Changes in w_{-q} that do not Affect the Equilibrium Price

The first system of equations is a simple consequence of the definition (1) of trader q 's equilibrium net trade as $x_q(w) = \chi_q(p(w), w_q)$. Trader q 's net trade $x_q(w)$ therefore remains constant as the endowment w_{-q} of all other traders is changed so long as the equilibrium price vector $p(w)$ remains constant. This principle is expressed from a first order perspective in the following theorem.

Theorem 1 *For each $1 \leq q \leq n$ and $1 \leq t \leq k$, trader q 's equilibrium net trade $x_{q,t}(w)$ in good t satisfies the equation*

$$\text{rank} \begin{pmatrix} D_{w_{-q}} p_{-k} \\ D_{w_{-q}} x_{q,t} \end{pmatrix} < k \quad (5)$$

on W .

Proof. Equation (5) holds trivially if $\text{rank } D_{w_{-q}} p_{-k} < k - 1$. Suppose instead that $\text{rank } D_{w_{-q}} p_{-k} = k - 1$ in a neighborhood of w^* . The null space of $D_{w_{-q}} p_{-k}(w^*)$ is the tangent space to the level set

$$\{w_{-q} \mid p(w_q^*, w_{-q}) = p(w^*)\} \subset W_{-q}$$

of $p(w_q^*, \cdot)$. Because $x_{q,t}(w_q^*, \cdot)$ is constant on this set, the row vector $D_{w_{-q}} x_{q,t}$ is linearly dependent with the rows of $D_{w_{-q}} p_{-k}(w^*)$, which implies (5). ■

The following argument shows that (5) holds for good k if it holds for goods $t < k$. Trader q 's budget equation $p(w) \cdot x_q(w) = 0$ implies

$$\sum_{t=1}^k x_{q,t} \cdot D_{w_{-q}} p_t + \sum_{t=1}^k p_t \cdot D_{w_{-q}} x_{q,t} = 0. \quad (6)$$

The normalization of prices to Δ allows $D_{w_{-q}} p_k$ to be expressed as a linear combination of the rows of $D_{w_{-q}} p_{-k}$. Substitution into (6) together with $p_k \neq 0$ then allows $D_{w_{-q}} x_{q,k}$ to be written as a linear combination of the rows of $D_{w_{-q}} p_{-k}$ and the $k-1$ vectors $D_{w_{-q}} x_{q,1}, \dots, D_{w_{-q}} x_{q,k-1}$. Therefore, $n(k-1)$ independent rank conditions are given by (5).

4 The Change in $\mathcal{E}(w)$ From a Change in w in the Direction of a Balanced and Feasible Net Trade

Let $\mathcal{E}(w^*) = (p^*, x^*)$. Let $N(p^*)$ denote the set of individually feasible and balanced net trade vectors given the price vector p^* ,

$$N(p^*) = \left\{ m = (m_i)_{1 \leq i \leq n} \in \mathbb{R}^{kn} \mid p^* \cdot x_i = 0 \text{ for each } i, \sum_{i=1}^n x_i = 0 \right\}.$$

Consider $m \in N(p^*)$ sufficiently small that $w^* + m \in W$. For each trader i , the budget line in \mathbb{R}_+^k determined by p^* and the endowment $w_i^* + m_i$ is the same as the budget line determined by p^* and the endowment w_i^* . The bundle $w_i^* + x_i^*$ is therefore optimal for trader i regardless of whether w_i^* or $w_i^* + m_i$ is his endowment. Because m is balanced, the price vector p^* and the net trade vector $x^* - m$ is therefore a competitive equilibrium for the endowment $w^* + m$.

Assumption 1. in section 2 asserts that $\mathcal{E}(w^* + m)$ is the unique competitive equilibrium for $w^* + m$ whose price vector is in $O \subsetneq \Delta$. Because $p^* \in O$, it follows that⁶

$$x(w^* + m) = x(w^*) - m \text{ and} \quad (7)$$

$$p(w^* + m) = p(w^*) = p^* \quad (8)$$

whenever $m \in N(p^*)$ and $w^* + m \in W$.

The system of equations derived in this section expresses (7) and (8) in terms of the derivatives of $p(w)$ and $x(w)$, which allows the variable m to be eliminated. For $t \in \mathbb{N}$, let I^t denote the $t \times t$ identity matrix. Because w^* is arbitrary, equations (7) and (8) are equivalent to

$$\begin{aligned} D_w x(w^*) \cdot m &= -m \Leftrightarrow \\ (D_w x(w^*) + I^{kn}) \cdot m &= 0 \end{aligned} \quad (9)$$

and

$$D_w p(w^*) \cdot m = 0 \quad (10)$$

holding at each $w^* \in W$ for $m \in N(p^*) = N(p(w^*))$. Let $\varepsilon_{i,t}$ denote the standard basis vector in \mathbb{R}^{kn} corresponding to trader i 's endowment $w_{i,t}$ of good t . Adding $\varepsilon_{i,t}$ to a vector thus increases the value of $w_{i,t}$ by 1. In terms of a trader's net trade in a particular good, (9) is equivalent to

$$(\nabla x_{q,t}(w^*) + \varepsilon_{q,t}) \cdot m = 0 \quad (11)$$

holding for each trader q and each good t , while (10) expressed in terms of individual prices is equivalent to

$$\nabla p_t(w^*) \cdot m = 0 \quad (12)$$

holding for each good t .

Equations (11) and (12) are equivalent to each $\nabla x_{q,t}(w^*)$ and $\nabla p_t(w^*)$ being linearly dependent with row vectors representing the linear system that defines

⁶This is the only point in the paper at which the "local uniqueness" of the competitive equilibrium price vector is needed, as assumed in 1. of section 2. Aside from Theorem 2 below, all other results require only that $p(w)$ is a C^2 selection of a competitive equilibrium price vector for $w \in W$.

$N(p^*)$. The first step to eliminating the variable m is to represent the linear equations

$$p^* \cdot m_i = 0 \text{ for each } i \text{ and } \sum_{i=1}^n m_i = 0 \quad (13)$$

that define $N(p^*)$ in matrix form. Because $p^* \cdot m_n = 0$ is implied by the other equations in the system (13), the reduced system of $n+k-1$ independent equations

$$\sum_{i=1}^n m_i = 0 \quad (14)$$

and

$$p^* \cdot m_i = 0 \text{ for } i < n \quad (15)$$

that defines $N(p^*)$ is considered below.

Let $Z^{t,s}$ be the $t \times s$ zero matrix. For traders $i < n$ let $P_i(w)$ denote the $(n+k-1) \times k$ -matrix

$$P_i(w) = \begin{pmatrix} I^k \\ Z^{i-1,k} \\ p(w) \\ Z^{n-i-1,k} \end{pmatrix},$$

and let

$$P_n(w) = \begin{pmatrix} I^k \\ Z^{n-1,k} \end{pmatrix}.$$

The system (14)-(15) is represented in matrix form by the $(n+k-1) \times nk$ -matrix

$$P(w^*) = (P_1(w^*) \quad \dots \quad P_n(w^*)).$$

The t th of first k rows represents the equation defined by $\sum_{i=1}^n m_i = 0$ on the t th good and the $(k+i)$ th row represents the budget equation $p^* \cdot m_i = 0$ of trader $i < n$. Because $p^* \neq 0$, it is clear that $P(w^*)$ has rank equal to $n+k-1$. The dependence of the equations (11) and (12) with this system implies the following theorem.

Theorem 2 *The rank conditions*

$$\text{rank} \begin{pmatrix} P(w) \\ \nabla p_l(w) \end{pmatrix} = n+k-1 \text{ and} \quad (16)$$

$$\text{rank} \begin{pmatrix} P(w) \\ \nabla x_{q,t}(w) + \varepsilon_{q,t} \end{pmatrix} = n + k - 1 \quad (17)$$

hold for each trader q , any goods t and l and at each $w \in W$.

The remainder of this section is devoted to eliminating redundancy in (16) and (17), first with respect to standard equations satisfied by a competitive equilibrium, and then with respect to system (5) of Theorem 1.

Independent Rank Conditions. The balance equation (2), the normalization of prices to the price simplex Δ , and the n budget equations (3) all imply redundancy in the rank conditions (16)-(17). The following discussion will show that there is a total of $n(k-1)$ rank conditions determined by (16) and (17) that are mutually independent and independent of these familiar equations on competitive equilibrium. Addressing (16) is straightforward: the normalization of prices to Δ implies that (16) holds for good k if it holds for goods $l < k$, which means that (16) determines $k-1$ independent rank conditions.

Turning to (17), the following argument shows that it holds in the case of trader $q = n$ if it holds for traders $q < n$. The first line below follows from the balance equation (2):

$$\text{rank} \begin{pmatrix} P(w^*) \\ \nabla x_{n,t}(w^*) + \varepsilon_{n,t} \end{pmatrix} = \text{rank} \begin{pmatrix} P(w^*) \\ -\sum_{q < n} \nabla x_{q,t}(w^*) + \varepsilon_{n,t} \end{pmatrix} \quad (18)$$

$$= \text{rank} \begin{pmatrix} P(w^*) \\ \sum_{q=1}^n \varepsilon_{q,t} \end{pmatrix}. \quad (19)$$

If (17) holds for traders $q < n$, then row operations on the second matrix in (18) implies that each $-\nabla x_{q,t}(w^*)$ can be replaced with $\varepsilon_{q,t}$, which implies the second equality above. The last row of the matrix (19) is the same as its t th row, which establishes the desired result. Trader q 's budget equation $p(w) \cdot x_q(w) = 0$ implies

$$\sum_{t=1}^k p_t \nabla x_{q,t} + \sum_{t=1}^k x_{q,t} \nabla p_t = 0. \quad (20)$$

A similar argument using (20) shows that (17) holds for good $t = k$ if (i) it holds for goods $t < k$ and (ii) the rank condition (16) holds for each price function $p_t(w)$. By varying q and t , there are thus a total of $(n-1)(k-1)$ rank conditions defined by (17) that are mutually independent, independent of the $k-1$ rank conditions given by (16), and also independent of the budget equations, price normalization, and the balance of net trades.

Redundancy with Theorem 1. The system of equations (5) rests upon the observation that $x_q(w)$ remains constant as $w_{-q} \in W_{-q}$ is varied so that $p(w)$ remains constant. The overlap between system (17) and system (5) concerns $m \in N(p^*)$ such that $m_q = 0$ (i.e., only the endowments of traders other than trader q are varied). Such a value of m changes w_{-q} in such a way that the equilibrium price p^* does not change. Systems (17) and (5) both express the observation that $x_q(w+m)$ is constant for such m . System (17) is reformulated in the following corollary to eliminate this redundancy between the two systems.

Corollary 3 *Given the systems (5) and (16) together with the normalization of prices, balance of net trades, and the n traders' budget equations, the rank condition (17) holding for each trader q and each good t reduces to the rank condition*

$$\text{rank} \begin{pmatrix} I^k & I^k \\ p(w) & Z^{1,k} \\ D_{w_q} x_{q,t}(w) + \varepsilon_{q,t} & D_{w_{q+1}} x_{q,t}(w) \end{pmatrix} = k + 1 \quad (21)$$

holding for $1 \leq q \leq n - 1$ and $1 \leq t \leq k - 1$.

The proof is in the Appendix.

Summary. To summarize the reduction of the systems (16)-(17), the normalization of prices to Δ implies that condition (16) on $p(w)$ reduces to $k - 1$ independent rank conditions. Price normalization and the budget and balance equations reduce (17) to $(n - 1)(k - 1)$ independent rank conditions. Taking into account system (5), (17) reduces to the $(n - 1)(k - 1)$ rank conditions of (21). There is no reduction in the number of rank conditions in reaching (21); a rank condition of (21), however, concerns a smaller matrix than a rank condition of (17), and hence is equivalent to a smaller number of independent determinant equations. This is the value of replacing (17) with (21). In all, $n(k - 1)$ independent rank conditions have been derived in this section.

5 Integrability Conditions on Prices

For the sake of interpretation, assume now that the price vector p is publicly observable while each trader privately observes his own endowment w_i . In the competitive mechanism, each trader i communicates his desired net trade $\chi_i(p, w_i)$. The price vector $p(w)$ solves the system of k market-clearing equations

$$\sum_{i=1}^n \chi_i(p, w_i) = 0 \quad (22)$$

at every $w \in W$. As a system of equations in w and p , (22) has two features that reflect the decentralization of the competitive mechanism: (i) trader i 's

endowment w_i affects the solution p solely through his excess demand $\chi_i(p, w_i)$; (ii) the excess demand functions for each of the k goods that make up $\chi_i(p, w_i)$ are functionally dependent by virtue of trader i 's budget constraint. Property (i) reflects the *dispersal* of information across traders in the sense that each trader i selects his own net trade knowing his own preferences and his budget constraint determined by p and w_i but without knowing the preferences and endowments of others. Property (ii) represents a savings in communication in the sense that trader i 's k -dimensional vector w_i is *condensed* into a $k - 1$ dimensional signal $\chi_i(p, w_i)$. This is not as striking as the condensation that occurs when a trader's preference information is modeled as a variable of large dimension and yet he still communicates using the $k - 1$ dimensional signal $\chi_i(p, w_i)$. It is significant, however, in the sense that it along with property (i) constrain the mappings $p(w)$ that can be realized by the competitive mechanism. Using properties (i) and (ii), a system of rank conditions is derived in this section that a mapping $p : W \rightarrow \Delta$ must satisfy in order to be realized by the competitive mechanism. The excess demand mappings $\chi_i(p, w_i)$ are essentially variables that are eliminated in solving for the rank conditions.

Again, let $\mathcal{E}(w^*) = (p^*, x^*)$. Define $S_i(w^*)$ as the level set of trader i 's excess demand $\chi_i(\cdot)$ through w_i^* taking p^* as fixed,

$$S_i(w^*) = \{w_i | \chi_i(p^*, w_i) = x_i^*\}. \quad (23)$$

The dependence of $S_i(w^*)$ upon w_{-i}^* is through the price vector $p^* = p(w^*)$. Notice that

$$\begin{aligned} \mathcal{E}^{-1}(p^*, x^*) &= \{w | \chi_i(p^*, w_i) = x_i^*, 1 \leq i \leq n\} \\ &= \{w | w_i \in S_i(w^*), 1 \leq i \leq n\} \\ &= \prod_{i=1}^n S_i(w^*). \end{aligned} \quad (24)$$

Equation (24) states that each level set of the competitive equilibrium mapping $\mathcal{E}(w)$ is a Cartesian product of n sets, the i th of which lies within W_i . This reflects property (i) above.

That each $S_i(w^*)$ contains other endowments w_i besides w_i^* reflects property (ii). The regularity condition (4) is now imposed to insure that each set $S_i(w^*)$ is a 1-dimensional C^2 submanifold of W_i . This follows from the equality

$$S_i(w^*) = \{w_i | p(w_i, w_{-i}^*) = p^*\}, \quad (25)$$

which is true by the following argument. First, (24) shows that $p(w_i, w_{-i}^*) = p^*$ for $w_i \in S_i(w^*)$, and so the left side of (25) is contained in the right side. Second,

$$x_j(w_i, w_{-i}^*) = \chi_j(p(w_i, w_{-i}^*), w_j^*) = x_j(w^*)$$

holds for any trader $j \neq i$ if $p(w_i, w_{-i}^*) = p^*$, and hence

$$x_i(w_i, w_{-i}^*) = \chi_i(p(w_i, w_{-i}^*), w_i) = x_i(w^*)$$

holds for such w_i by the balance of net trades. The right side of (25) is thus contained in the left side.

Assuming the regularity condition (4), each trader i 's excess demand function thus defines a C^2 foliation⁷ of the space W of endowments,

$$\mathcal{S}_i = \{S_i(w) \times \{w_{-i}\} | w \in W\},$$

where each $S_i(w) \times \{w_{-i}\}$ is a 1-dimensional C^2 submanifold of W . Let \mathcal{S} denote the foliation of W defined by the level sets of $\mathcal{E}(w)$,

$$\mathcal{S} = \{\mathcal{E}^{-1}(p(w), x(w)) | w \in W\}.$$

The system of equations below states differential conditions on $p(\cdot)$ for the foliations $\mathcal{S}_1, \dots, \mathcal{S}_n$ and \mathcal{S} to satisfy (24) at every $w^* \in W$. The system is obtained by considering the normal direction to the n -dimensional manifolds of \mathcal{S} , which by (25) is spanned by the vectors in the set

$$\{D_{w_i} p_l | 1 \leq i \leq n, 1 \leq l \leq k-1\}.$$

The Frobenius Theorem states necessary and sufficient conditions on a spanning set of the normal direction for the existence of the corresponding foliation. The equations in the theorem below present these conditions in a reduced form.⁸

Some notation is needed to state the theorem. For any good l and for distinct traders i and j , let $D_{w_i, w_j}^2 p_l$ denote the $k \times k$ matrix of mixed partials

$$D_{w_i, w_j}^2 p_l = \left(\frac{\partial^2 p_l}{\partial w_{i,t} \partial w_{j,s}} \right)_{1 \leq t, s \leq k}.$$

Let Z^t denote the $t \times t$ zero matrix for any $t \in \mathbb{N}$. The equations on $p(w)$ concern the *bordered mixed Hessian* $H_{i,j}(p, p_l)$, which is the $(2k-1) \times (2k-1)$ matrix⁹

$$H_{i,j}(p, p_l) = \begin{pmatrix} Z^{k-1} & D_{w_j} p_{-k} \\ (D_{w_i} p_{-k})^T & D_{w_i, w_j}^2 p_l \end{pmatrix}. \quad (26)$$

Theorem 4 *A competitive equilibrium price mapping $p(w)$ satisfies*

$$\text{rank } H_{i,j}(p, p_l) < 2k-1 \quad (27)$$

for any good l and for each pair of distinct traders i and j .

⁷A C^2 foliation of an open set $Z \subset \mathbb{R}^u$ is a partition $\{T(z) | z \in Z\}$ of Z such that (i) each set $T(z)$ is a C^2 submanifold of Z of dimension d for some $d \in \mathbb{N}$, and (ii) the submanifolds in the partition vary smoothly in the sense that there exists a set of d C^1 vector fields on Z that span the tangent space to $T(z)$ at every $z \in Z$.

⁸The mathematical approach of this section originates in Hurwicz, Reiter and Saari (1978, Ch. IV) and Saari (1984) as part of their effort to develop a differential approach to decentralization. See also Williams (2002) for a more comprehensive development of this theory.

⁹The bordered Hessian form of the equations in the following theorem are inspired by the bounds on minimal message space dimension derived in Hurwicz (1979), Williams (1984), and Chen (1992).

Theorem 4 is proven in the Appendix. Though it is used above, the regularity condition (4) on $p(\cdot)$ is not required as a hypothesis in this theorem, essentially because equation (27) holds trivially at a given w^* if either $D_{w_i}p_{-k}$ or $D_{w_j}p_{-k}$ has rank less than $k - 1$ at this endowment.

Interchanging the role of i and j in (27) does not change this equation, and so each pair of distinct traders i and j determines exactly one equation for each good l . It is also pointed out in the proof that $p(\cdot)$ satisfies (27) in the case of $l = k$ if it satisfies (27) for all $l < k$. The rank condition (27) is equivalent to $\det H_{i,j}(p, p_l) = 0$. By varying i, j and l , (27) thus determines a system of $(k - 1)n(n - 1)/2$ independent equations on the first and second derivatives of $p(w)$, with the $n(n - 1)/2$ term counting the number of sets $\{i, j\}$ with $i \neq j$.

6 Example: $k=2$ Goods, $n=2$ Traders with Cobb-Douglas Utility Functions

Preferences for the traders are not specified in this paper and they do not appear in the three systems that are presented in Theorems 1-3. They are effectively variables that are implicit in the equilibrium mapping $\mathcal{E}(w)$ and in the traders' excess demand mappings but that are eliminated in deriving the three systems. It is instructive, however, to verify that each of the three systems holds in the case of particular utility functions for which $p(w)$ can be explicitly computed. This is accomplished here using a Cobb-Douglas utility function for each trader. Rather than normalizing prices to Δ , it is assumed in this example that good 2 is a numéraire. The equilibrium price of good 1 is denoted simply as $p(w)$ below instead of $p_1(w)$.

Letting $y_i = (y_{i,1}, y_{i,2})$ represent an allocation for trader i , the utility functions of traders 1 and 2 are

$$u_1(y_{1,1}, y_{1,2}) = y_{1,1}^\delta y_{1,2}^{1-\delta} \text{ and}$$

$$u_2(y_{2,1}, y_{2,2}) = y_{2,1}^\gamma y_{2,2}^{1-\gamma},$$

respectively, for fixed values of δ, γ in $(0, 1)$. A property of this family of utility functions is that the exponent of $y_{i,1}$ is the proportion of trader i 's income that he spends on good 1. Consequently,

$$py_{1,1} = \delta(pw_{1,1} + w_{1,2}) \text{ and} \tag{28}$$

$$py_{2,1} = \gamma(pw_{2,1} + w_{2,2}),$$

where $y_{i,1}$ now represents trader i 's demand for good 1 at the price p . Summing these two equations produces

$$p(y_{1,1} + y_{2,1}) = p(\delta w_{1,1} + \gamma w_{2,1}) + (\delta w_{1,2} + \gamma w_{2,2}). \tag{29}$$

Feasibility of competitive equilibrium requires that $y_{1,1} + y_{2,1} = w_{1,1} + w_{2,1}$. Substitution into (29) and simplifying implies

$$p[(1 - \delta)w_{1,1} + (1 - \gamma)w_{2,1}] = \delta w_{1,2} + \gamma w_{2,2},$$

and so

$$p(w) = \frac{\delta w_{1,2} + \gamma w_{2,2}}{(1 - \delta)w_{1,1} + (1 - \gamma)w_{2,1}}. \quad (30)$$

Each trader's equilibrium net trade of good 1 is also needed below. From (28) it is clear that

$$\begin{aligned} x_{1,1}(w) &\equiv y_{1,1} - w_{1,1} \\ &= \frac{\delta(p(w)w_{1,1} + w_{1,2})}{p(w)} - w_{1,1} \\ &= -(1 - \delta)w_{1,1} + \frac{\delta w_{1,2}}{p(w)}, \end{aligned} \quad (31)$$

and similarly,

$$x_{2,1}(w) = -(1 - \gamma)w_{2,1} + \frac{\gamma w_{2,2}}{p(w)}. \quad (32)$$

The Rank Condition (5) of Theorem 1. The rank condition (5) in this case reduces to

$$\text{rank} \begin{pmatrix} D_{w_2} p \\ D_{w_2} x_{1,1} \end{pmatrix} < 2 \quad (33)$$

and

$$\text{rank} \begin{pmatrix} D_{w_1} p \\ D_{w_1} x_{2,1} \end{pmatrix} < 2. \quad (34)$$

Substitution of (31) into (33) produces

$$\begin{pmatrix} \frac{\partial p}{\partial w_{2,1}} & \frac{\partial p}{\partial w_{2,2}} \\ -\frac{\delta w_{1,2}}{p(w)^2} \frac{\partial p}{\partial w_{2,1}} & -\frac{\delta w_{1,2}}{p(w)^2} \frac{\partial p}{\partial w_{2,2}} \end{pmatrix},$$

and so (33) obviously holds. Substitution of (32) into (34) produces

$$\begin{pmatrix} \frac{\partial p}{\partial w_{1,1}} & \frac{\partial p}{\partial w_{1,2}} \\ -\frac{\gamma w_{2,2}}{p(w)^2} \frac{\partial p}{\partial w_{1,1}} & -\frac{\gamma w_{2,2}}{p(w)^2} \frac{\partial p}{\partial w_{1,2}} \end{pmatrix},$$

and so it also holds.

The Rank Conditions (16) and (17) of Theorem 2. The matrix in the rank condition (16) in this case is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ p(w) & 1 & 0 & 0 \\ \frac{\partial p}{\partial w_{1,1}} & \frac{\partial p}{\partial w_{1,2}} & \frac{\partial p}{\partial w_{2,1}} & \frac{\partial p}{\partial w_{2,2}} \end{pmatrix}. \quad (35)$$

The 3×3 submatrix of (35) defined by its first three rows and columns has nonzero determinant and so (35) has rank at least 3. To confirm (16), it is therefore sufficient to show that the determinant of (35) is zero. Expanding along its first row, this determinant equals

$$\left(\frac{\partial p}{\partial w_{2,1}} \right) + \left(-p(w) \frac{\partial p}{\partial w_{2,2}} + p(w) \frac{\partial p}{\partial w_{1,2}} - \frac{\partial p}{\partial w_{1,1}} \right).$$

This reduces to

$$\frac{-(1-\gamma)p(w) - \gamma p(w) + \delta p(w) + (1-\delta)p(w)}{(1-\delta)w_{1,1} + (1-\gamma)w_{2,1}} = 0$$

for $p(w)$ defined in (30), which completes the verification.

As discussed after Theorem 2, the rank condition (17) by varying the trader q and the good t defines $(n-1)(k-1)$ independent rank conditions. Condition (17) is thus verified in this example by considering $q = k = 1$. Applying formula (31) for $x_{1,1}(w)$, the matrix in (17) in this case is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ p(w) & 1 & 0 & 0 \\ \frac{\partial x_{1,1}}{\partial w_{1,1}} + 1 & \frac{\partial x_{1,1}}{\partial w_{1,2}} & \frac{\partial x_{1,1}}{\partial w_{2,1}} & \frac{\partial x_{1,1}}{\partial w_{2,2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ p(w) & 1 & 0 & 0 \\ \delta - \frac{\delta w_{1,2}}{p(w)^2} \frac{\partial p}{\partial w_{1,1}} & \frac{\delta}{p(w)} - \frac{\delta w_{1,2}}{p(w)^2} \frac{\partial p}{\partial w_{1,2}} & -\frac{\delta w_{1,2}}{p(w)^2} \frac{\partial p}{\partial w_{2,1}} & -\frac{\delta w_{1,2}}{p(w)^2} \frac{\partial p}{\partial w_{2,2}} \end{pmatrix}.$$

Multiplying row 4 by $-p(w)^2/\delta w_{1,2}$ and then adding $p(w)/w_{1,2}$ times row 3 to row 4 reduces this last matrix to the matrix (35), which is shown above to have rank equal to 3. This completes the verification of (17).

The Integrability Condition (27) of Theorem 4. For notational convenience, define

$$B(w_{1,1}, w_{2,1}) \equiv (1 - \delta) w_{1,1} + (1 - \gamma) w_{2,1}.$$

The matrix in (27) equals

$$\frac{1}{B(w_{1,1}, w_{2,1})^2} \cdot \begin{pmatrix} 0 & -(1 - \gamma)(\delta w_{1,2} + \gamma w_{2,2}) & \gamma B(w_{1,1}, w_{2,1}) \\ -(1 - \delta)(\delta w_{1,2} + \gamma w_{2,2}) & 2(1 - \delta)(1 - \gamma)p(w) & -\gamma(1 - \delta) \\ \delta B(w_{1,1}, w_{2,1}) & -\delta(1 - \gamma) & 0 \end{pmatrix}.$$

Consider the following sequence of operations on the above matrix: (i) multiply column 3 by $1/\gamma$; multiply column 2 by $1/(1 - \gamma)$; multiply row 2 by $1/(1 - \delta)$; multiply row 3 by $1/\delta$; add $1/B(w_{1,1}, w_{2,1})$ times column 1 to column 2; add $1/B(w_{1,1}, w_{2,1})$ times row 1 to row 2. This sequence of row and column operations reduces this matrix to

$$\begin{pmatrix} 0 & -(\delta w_{1,2} + \gamma w_{2,2}) & (1 - \delta) w_{1,1} + (1 - \gamma) w_{2,1} \\ -(\delta w_{1,2} + \gamma w_{2,2}) & 0 & 0 \\ (1 - \delta) w_{1,1} + (1 - \gamma) w_{2,1} & 0 & 0 \end{pmatrix}.$$

This matrix clearly has rank equal to 2, and so the price vector $p(w)$ satisfies (27).

7 Conclusion

The systems of rank conditions derived in this paper are rooted in two properties of the traders' excess demand mappings:

1. Reflecting the *decentralization* of decision-making in the competitive mechanism, each trader i 's excess demand $\chi_i(p, w_i)$ is affected by the endowments of other traders only insofar as they affect the price vector p that he faces.
2. Reflecting the *optimality* of each trader i 's excess demand $\chi_i(p, w_i)$ subject to his budget constraint given by the price vector p and his endowment w_i , the equation

$$\chi_i(p, w_i + m_i) = \chi_i(p, w_i) - m_i$$

holds for any sufficiently small net trade vector m_i that satisfies $p \cdot m_i = 0$.

Theorem 1 states the rather obvious consequences of 1. for the equilibrium net trade mapping $x(w)$, while Theorem 4 states the less obvious consequences of this fact upon the equilibrium price vector $p(w)$. Theorem 2 states the implications of 2. for both $x(w)$ and $p(w)$.

Price normalization, balance of net trades, the traders' budget equations have been applied to simplify each of the three systems of rank conditions. Redundancy between two of the three systems is addressed in Corollary 3. While each system clearly restricts $x(\cdot)$ or $p(\cdot)$ in ways that the other two systems do not, no proof is provided of complete independence of the three systems from one another other. Regardless of the precise number of independent equations that are derived herein, it is clear from (16) and (27) that nontrivial equations on $p(\cdot)$ have been found that do not follow from price normalization, balance of net trades and the budget equations alone. These equations show that a competitive equilibrium price mapping $p(\cdot)$ is nongeneric among all C^2 mappings from W to Δ in the sense that it lies in a closed and nowhere dense subset of all such mappings in the Whitney C^2 topology. To my knowledge, this has not previously been noted.

The three systems of rank conditions on $p(w)$ and $x(w)$ pose at least two questions for future research. First, are the equations implied by these rank conditions empirically testable? This question ties this paper back to the research program discussed in the Introduction. Second, what additional conditions beyond these equations must a mapping $p(w)$ satisfy if it is to represent the equilibrium price mapping for some choice of the preferences for the n traders? This last question seeks sufficient conditions to insure that a mapping $p(w)$ represents a particular market aggregate. The results of this paper are opposite in theme from the Debreu-Mantel-Sonnenschein Theorem in that they identify structure in two market aggregates. This last question, however, shares the motivation of this famed result in the sense that both address sufficient conditions under which a given mapping represents a particular market aggregate.

8 Appendix

Proof of Corollary 3. Let $N_{-q}(p^*)$ denote the subspace of $N(p^*)$ defined by $m_q = 0$. The key to eliminating the redundancy between systems (17) and (5) is to reformulate system (17) so that it only addresses perturbations m in a linear space $T_q(p^*)$ that satisfies

$$N(p^*) = T_q(p^*) \oplus N_{-q}(p^*).$$

Returning to (11), a system is now derived that expresses

$$(\nabla x_{q,t}(w^*) + \varepsilon_{q,t}) \cdot m = 0 \tag{36}$$

for m in such a space $T_q(p^*)$.

It is shown in the analysis of the system (17) that q and t can be restricted to $1 \leq q \leq n-1$ and $1 \leq t \leq k-1$. One of many possible choices for $T_q(p^*)$ is

$$T_q(p^*) = \{m \mid m \in N(p^*), m_j = 0 \text{ for } j \neq q, q+1\}.$$

For this choice of $T_q(p^*)$, (36) reduces to

$$(D_{w_q}x_{q,t}(w^*) + \varepsilon_{q,t}) \cdot m_q + D_{w_{q+1}}x_{q,t}(w^*) \cdot m_{q+1} = 0 \quad (37)$$

for $m_q, m_{q+1} \in \mathbb{R}^k$ such that

$$m_q + m_{q+1} = 0 \text{ and } p \cdot m_q = 0. \quad (38)$$

The remainder of the proof represents the system (37)-(38) as (21). The first k rows of the matrix in (21) represent $m_q + m_{q+1} = 0$ and row $k + 1$ represents $p(w) \cdot m_q = 0$. The first $k + 1$ rows therefore represent the system (38). Row $k + 2$ represents equation (37). Because $p(w) \neq 0$, this matrix has rank at least $k + 1$. The rank condition (21) follows because (37) holds for m_q, m_{q+1} satisfying (38). ■

Proof of Theorem 4. As before, let $\mathcal{E}(w^*) = (p^*, x^*)$. Assume for notational convenience that $i = 1$ and $j = 2$. The desired rank condition (27) holds trivially at w^* for these values of i and j if the regularity condition (4) on $p(\cdot)$ fails to hold for either trader 1 or trader 2 at the given point. It is thus assumed below that (4) holds for traders 1 and 2 in a neighborhood of w^* .

The analysis is now reduced to a two-trader problem by fixing the endowments of traders 3 through n at the values determined by w^* .¹⁰ Let $w_{3+}^* \in \mathbb{R}_{++}^{(n-2)k}$ denote this vector. The notation “ \sim ” is used below to indicate the restricted case obtained by specifying w_{3+}^* . Define $\tilde{p}(w_1, w_2)$ and $\tilde{\mathcal{E}}(w_1, w_2)$ to be the restrictions of $p(w)$ and $\mathcal{E}(w)$ to $W_1 \times W_2$ given w_{3+}^* ,

$$\tilde{p}(w_1, w_2) = p(w_1, w_2, w_{3+}^*), \text{ and}$$

$$\tilde{\mathcal{E}}(w_1, w_2) = \mathcal{E}(w_1, w_2, w_{3+}^*).$$

The mappings $\tilde{p}_l(w_1, w_2)$ and $\tilde{p}_{-k}(w_1, w_2)$ are defined similarly. It is shown below that

$$\text{rank } H_{1,2}(\tilde{p}, \tilde{p}_l) < 2k - 1, \quad (39)$$

which is exactly the rank condition (27) in the case of $i = 1$ and $j = 2$.

For $t = 1, 2$ define

$$\tilde{S}_t(w_1, w_2) = S_t(w_1, w_2, w_{3+}^*).$$

¹⁰The analysis that follows addresses the integrability of the foliation given by the sets $\tilde{S}_1(w) \times \tilde{S}_2(w)$ that are defined below rather than the foliation \mathcal{S} given by the sets $\prod_{i=1}^n S_i(w)$, as suggested before the statement of Theorem 4. This is for the sake of expediency in proving the theorem and also to address the possibility that the regularity condition (4) does not hold at w^* for all traders 3 through n . It is shown in Williams (2002, sec. 4.6.6) that (27) is a necessary and sufficient set of integrability conditions for the foliation \mathcal{S} when the regularity condition (4) holds for each trader.

The regularity condition (4) insures that each $\tilde{S}_t(w_1, w_2)$ is a 1-dimensional C^1 submanifold of W_t near w_t^* . Equation (24) is easily reformulated to show that if $\tilde{\mathcal{E}}(w'_1, w'_2) = (p', x')$, then

$$\tilde{\mathcal{E}}^{-1}(p', x') = \tilde{S}_1(w'_1, w'_2) \times \tilde{S}_2(w'_1, w'_2). \quad (40)$$

The set

$$\tilde{\mathcal{S}} = \left\{ \tilde{S}_1(w_1, w_2) \times \tilde{S}_2(w_1, w_2) \mid (w_1, w_2) \in W_1 \times W_2 \right\} \quad (41)$$

thus defines a foliation of $W_1 \times W_2$ near (w_1^*, w_2^*) . The remainder of the proof is devoted first to stating integrability conditions on p_{-k} for $\tilde{\mathcal{S}}$ to be a foliation and then to reducing these conditions to the simple form of (39).

The foliation $\tilde{\mathcal{S}}$ determines a differential ideal \mathcal{I} that is generated by the 1-forms in the set

$$\{d_1\tilde{p}_l \mid 1 \leq l \leq k-1\} \cup \{d_2\tilde{p}_l \mid 1 \leq l \leq k-1\} \quad (42)$$

where

$$d_t p_l = \sum_{u=1}^k \frac{\partial p_l}{\partial w_{t,u}} dw_{t,u}.$$

The set (42) generates \mathcal{I} because: (i) (40) implies that each $\tilde{p}_l(\cdot)$ is constant on each $\tilde{S}_t(w'_1, w'_2) \times \{w'_{-t}\}$ and so $d_t \tilde{p}_l \in \mathcal{I}$ for $t = 1, 2$ and $1 \leq l \leq k$; (ii) the submanifolds in $\tilde{\mathcal{S}}$ are of codimension $2(k-1)$ in $W_1 \times W_2$, which is the number of linearly independent 1-forms in (42).¹¹ Because \mathcal{I} defines a foliation $\tilde{\mathcal{S}}$ of $W_1 \times W_2$ near (w_1^*, w_2^*) , the Frobenius Theorem¹² implies that

$$d(d_t \tilde{p}_l) \in \mathcal{I} \quad (43)$$

for $t = 1, 2$ and $1 \leq l \leq k$, where $d(d_t \tilde{p}_l)$ is the standard differential of the 1-form $d_t \tilde{p}_l$. Notice that

$$-d(d_1 \tilde{p}_l) = d(d_2 \tilde{p}_l) = \sum_{1 \leq t, u \leq k} \frac{\partial^2 \tilde{p}_l}{\partial w_{1,t} \partial w_{2,u}} dw_{1,t} \wedge dw_{2,u}. \quad (44)$$

It is thus sufficient to consider (43) only in the case of $t = 2$.¹³

¹¹The identification of (42) as a set of 1-forms that generates \mathcal{I} follows from Theorem 7 of Hurwicz, Reiter and Saari (1978) or Theorem 2.2 of Saari (1984). This step along with the integrability condition (43) together form the first order conditions for the existence of $\tilde{\mathcal{S}}$.

¹²A statement of the Frobenius Theorem in terms of differential ideals can be found in Spivak (1979, Prop. 14, p. 293) or Warner (1971, Prop. 2.30, p. 74).

¹³The case of $l = k$ is included in the derivation that follows. Notice that (43) holds for $l = k$ if it holds for $l < k$ because of price normalization, which implies $d_t \tilde{p}_k = -\sum_{l < k} d_t \tilde{p}_l$. The rank condition (27) that is derived below from (43) thus holds for $l = k$ if it holds for $l < k$.

The generating set (42) of \mathcal{I} implies that (43) holds only if there exists C^0 1-forms

$$\{\tau_s, \rho_s \mid 1 \leq s \leq k-1\}$$

such that

$$d(d_2\tilde{p}_l) = \sum_{s=1}^{k-1} \tau_s \wedge d_1p_s + \sum_{s=1}^{k-1} \rho_s \wedge d_2p_s. \quad (45)$$

Let Ψ_t denote the 1-forms spanned by

$$\{dw_{t,u} \mid 1 \leq u \leq k\}.$$

Because every term in the expansion (44) of $d(d_2\tilde{p}_l)$ is a wedge product of an element of Ψ_1 with an element of Ψ_2 , it can be assumed without loss of generality that each $\tau_s \in \Psi_2$ and each $\rho_s \in \Psi_1$. All other terms in (45) necessarily cancel.

It is now shown that equation (39) is necessary and sufficient for the existence of 1-forms that satisfy (45). This is apparent from noting that row operations on $H_{1,2}(\tilde{p}, \tilde{p}_l)$ involving the rows of D_{w_2p-k} and $D_{w_1, w_2}^2 p_l$ correspond to reducing $d(d_2\tilde{p}_l)$ using the 2-forms in the set

$$\{dw_{1,t} \wedge d_2p_s \mid 1 \leq s \leq k-1, 1 \leq t \leq k\},$$

while column operations involving the columns of $(D_{w_1p-k})^T$ and $D_{w_1, w_2}^2 p_l$ correspond to reducing $d(d_2\tilde{p}_l)$ using the 2-forms in the set

$$\{dw_{2,t} \wedge d_1p_s \mid 1 \leq s \leq k-1, 1 \leq t \leq k\}.$$

Subtracting the term $\rho_s \wedge d_2p_s$ in (45) from $d(d_2\tilde{p}_l)$ is thus equivalent to a sequence of row operations on $H_{1,2}(\tilde{p}, \tilde{p}_l)$ involving $D_{w_1, w_2}^2 p_l$ and row s of D_{w_2p-k} while subtracting the term $\tau_s \wedge d_1p_s$ from $d(d_2\tilde{p}_l)$ is equivalent to a sequence of column operations on $H_{1,2}(\tilde{p}, \tilde{p}_l)$ involving $D_{w_1, w_2}^2 p_l$ and column s of $(D_{w_1p-k})^T$. Equation (45) therefore states that $H_{1,2}(\tilde{p}, \tilde{p}_l)$ can be reduced through row and column operations to

$$\begin{pmatrix} Z^{k-1} & D_{w_2\tilde{p}-k} \\ (D_{w_1\tilde{p}-k})^T & Z^k \end{pmatrix}.$$

Given the regularity condition (4), this is equivalent to

$$\text{rank } H_{1,2}(\tilde{p}, \tilde{p}_l) = 2k - 2,$$

which implies (39). ■

References

- [1] Brown, D. J., and R. L. Matzkin (1996): “Testable Restrictions on the Equilibrium Manifold,” *Econometrica* 64, 1249-1262.
- [2] Chen, P. (1992): “A Lower Bound for the Dimension of the Message Space of the Decentralized Mechanisms Realizing a Given Goal,” *Journal of Mathematical Economics* 21, 249-270.
- [3] Chiappori, P. A., I. Ekelund, F. Kübler, and H. M. Polemarchakis (2002): “Testable Implications of General Equilibrium Theory: a Differentiable Approach,” mimeo., Brown University.
- [4] Debreu, G. (1974): “Excess Demand Functions,” *Journal of Mathematical Economics* 1, 15-23.
- [5] Hurwicz, L. (1979): “An Explicit Analysis of the Case Where Each of the Two Agents has Three Parameters ($N = 2, k_1 = k_2 = k = 3$),” mimeo., University of Minnesota.
- [6] Hurwicz, L., S. Reiter, and D. G. Saari (1978): “On Constructing Mechanisms with Message Spaces of Minimal Dimension for Smooth Performance Functions,” mimeo., University of Minnesota.
- [7] Mantel, R. (1974): “On the Characterization of Aggregate Excess Demand,” *Journal of Economic Theory* 7, 348-353.
- [8] Mas-Colell, A. (1977): “On the Equilibrium Price Set of an Exchange Economy,” *Journal of Mathematical Economics* 4, 117-126.
- [9] Nachbar, J. H. (2001a): “General Equilibrium Comparative Statics,” mimeo., Washington University (forthcoming in *Econometrica*).
- [10] Nachbar, J. H. (2001b): “General Equilibrium Comparative Statics: Discrete Shocks in Production Economies” mimeo., Washington University.
- [11] Saari, D. G. (1984): “A Method for Constructing Message Systems for Smooth Performance Functions,” *Journal of Economic Theory* 33, 249-274.
- [12] Sonnenschein, H. (1972): “Market Excess Demand Functions,” *Econometrica* 40, 549-563.
- [13] Sonnenschein, H. (1973): “Do Walras Identity and Continuity Characterize the Class of Community Excess Demand Functions,” *Journal of Economic Theory* 6, 345-354.
- [14] Spivak, M. (1979): *A Comprehensive Introduction to Differential Geometry, vol. I*. Berkeley: Publish or Perish.
- [15] Warner, F. (1971): *Foundations of Differentiable Manifolds and Lie Groups*. Glenview: Scott, Foresman and Company.

- [16] Williams, S. R. (1984): "Implementing a Generic Smooth Function," *Journal of Mathematical Economics* 13, 273-288.
- [17] Williams, S. R. (2002): "A Differential Approach to Economic Decentralization," mimeo., University of Illinois.