The Bayesian Newsvendors in Supply Chains with Unobserved Lost Sales

David Glenn
*University of Illinois at Urbana−Champaign*

Arnab Bisi
*University of British Columbia*

Martin L. Puterman
*University of British Columbia*

**Abstract**

We consider two−echelon supply chains with one supplier and two retailers. Retailers are censored newsvendors facing general parametric demand distributions involving unknown parameters. Using a Bayesian MDP formulation, we investigate how the supplier can make use of the combined information gathered from the retailers’ sales data to increase channel profits. We compare among the following three scenarios: (i) supplier shares her (pooled) demand distribution updates with the retailers, (ii) supplier does not use pooled updates, and (iii) integrated rm. We show that buy−back contracts achieve channel coordination in the above system.

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Arnab Bisi        David Glenn        Martin L. Puterman

Faculty of Commerce and Business Administration
University of British Columbia
2053 Main Mall
Vancouver, B.C., Canada V6T 1Z2

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Key words: Supply chain. Inventory. Bayesian Markov decision processes. Lost sales. Demand estimation. Censoring. Myopic policy.
Recent advancements in information technology have made it possible for the partner firms to share information in real time. There is a growing interest in designing effective supply chain systems by making the retailers’ demand and inventory data visible to the supplier. Most investigations in supply chain inventory management with stochastic demand assume that the demand distribution is known. However, this is rarely the case in practice and very often estimation of the demand distribution is required. The problem is further exacerbated if the demand observations get censored that occurs when there are lost sales and no backordering. For a two-echelon supply chain, here we study a mechanism of how to make use of all available demand information to determine the inventory levels of the retailers who face unknown and censored demand, the goal being to maximize the channel profits of the supply chain.

There is extensive research on finding and characterizing inventory policies in supply chains assuming that the retail demand distribution is known. Cachon and Fisher (2000) noted that for a known demand distribution, the observed benefits of implementing information technology are mainly due to reduced lead time and smaller batch size, not very much just because of sharing demand data to improve the supplier’s production scheduling. Chen (1998) found that information sharing lowered supply chain costs on average only by 1.8%. Aviv and Federgruen (1998) reported benefits of 0% - 5% for a vendor managed inventory (VMI) system. For an autoregressive AR(1) demand process with known parameters, Raghunathan (2001) showed that information sharing does not significantly improve supply chain performance. Graves (1999) found no benefit of information sharing for a similar nonstationary demand process. In this paper we demonstrate that for unknown demand distributions with unobserved lost sales, great benefits can be achieved if the information contained in multiple retailers’ sales data are properly pooled and utilized to determine the retail inventory levels. Although research on supply chain systems with unknown and censored demand data is scarce, there is a handful of literature available in similar context of the inventory systems. Conrad (1976) investigated the effect of demand censoring on Poisson demand estimation, and proposed an unbiased maximum likelihood estimate (MLE) of the Poisson parameter. Nahmias (1994) considered a censored demand model with normal data and proposes a procedure for sequentially updating estimates of the normal parameters. Agrawal and Smith (1996) suggested that the negative binomial distribution provides a better fit than either the normal or Poisson distributions to discrete data and developed a parameter estimation method for base stock inventory systems in which sales are truncated at a constant level. Ding (2001) developed Bayesian and maximum likelihood estimates of the parameters of Poisson and zero-inflated Poisson demand distributions in DCI and fully observable newsvendor and \((s,S)\) inventory systems. Her methods take into account of the fact that in an \((s,S)\) system the censoring level varies as the stock decreases during lead time.

When the demand distribution is not known with certainty, it is desirable to update the demand distribution as new data become available while avoiding storage of all historical data. One approach is to implement a Bayesian method. Karlin (1960), Scarf (1960) and Iglehart (1964) studied dynamic inventory policy updating when the demand density has unknown parameters and is a member of the exponential and range families. They showed that an adaptive critical value (or order-up-to) policy is optimal where the critical value depends on the history path through a sufficient statistic. Scarf (1960) and Iglehart (1964) also discussed the asymptotic behavior of the optimal adaptive policy. Azoury (1985) extended the work to more general demand distributions. Azoury (1984) also investigated the effect of dynamic Bayesian demand updating on optimal order quantities. She concluded that Bayesian demand updating, compared to non-Bayesian method, yields a more flexible optimal policy by allowing updates of the order quantities in future periods. Although the Bayesian approach provides a rigorous framework for dynamic demand updating, it is generally difficult to implement because of extensive computational demands. Lovejoy (1990) showed that a
simple inventory policy based on a critical fractile can be optimal or near-optimal in some inventory models. He also gave two numerical examples to illustrate the performance of the simple myopic policies. In the context of the design of a regulator for an adaptive controlled system with observable states, Florentin (1962) observed that the optimal control determined by Bayesian updating is higher than that of the non-Bayesian method. He assumed normal distribution for both the state variable and the unknown parameter. Silver and Fiechter (1995) used a Bayesian approach to study the problem of selecting a preventive maintenance interval for equipment that is subject to breakdown. They assumed that the operating time until breakdown has an unknown discrete distribution and its observations are censored by the chosen preventive maintenance interval.

In this paper, we consider a two-echelon supply chain with one supplier and two retailers. Retailers face demand distributions that involve unknown parameters. Moreover, demand observations get censored by the inventory level on hand. Supplier can see the retailers’ point-of-sale (POS) data in each period. To investigate the benefits of information sharing we use a Bayesian Markov decision process (BMDP) (van Hee (1978)) formulation which allows demand distribution updating and policy updating as information is gathered. However, a serious disadvantage of Bayesian MDP model is its probability distribution valued state variable and computational intractability. To reduce this complexity and capture the essence of the problem we assume that the retailers are newsvendors. While the model is conceptually simple, it has practical significance in the fashion goods and high-tech industries like computer manufacturing where products are characterized by rapid obsolescence, volatile markets, large lot sizes and long lead times. Eppen and Iyer (1997) described a related inventory problem facing merchandising managers in fashion industry in which managers have the option of dumping a portion of stocks before the end of the season. They used a newsvendor heuristic to simplify computation.

We study how the supplier can make use of the combined information gathered from two retailers’ sales data to increase the profits of the entire supply chain as well as some or all of the channel members. Our main objective is to compare the performances of the following three scenarios in our system: (i) supplier shares her (pooled) demand distribution updates with the retailers, (ii) supplier does not use pooled updates, and (iii) the integrated firm. We also show that the combined effect of an unknown demand distribution and unobservable lost sales results in higher optimal inventory levels for the retailers than that would be determined using a Bayesian myopic policy. That is, it is optimal to stock more so that the censoring (inventory) level is higher in early periods. By doing this, all the managers in the system acquire additional information about the demand distribution that can lead to better decisions in later periods. Lariviere and Porteus (1999) pursued similar direction of research for the specific case of an exponential demand distribution with a gamma conjugate prior in the context of a inventory system in multiple markets. In our paper, using a BMDP formulation we generalize the work to accommodate any parametric demand distribution with any prior distribution (not necessarily conjugate). We believe that this paper might be the first one in supply chain inventory management which shows, for uncertain and censored demand, how to increase supply chain profits by making retail inventory decisions based on the updated demand distribution that incorporates multiple sales data in each period. It also explicitly illustrates and interprets the trade off between information and optimality in a very general setup. Furthermore, our model is of collaborative forecasting (CF) type since the supplier updates the demand distribution based on all retailers’ sales data. Nowadays there is a lot of interest in studying CF techniques. Fisher and Raman (1996) showed that significant profits could be obtained for fashion products if one updates the demand forecasts based on early sales data and implement Quick Response (QR) programs in the system. It is reasonable to guess that as QR programs become more popular, Bayesian approach would be worth to pursue as it might be able to provide valuable solutions to many problems, specially when either no historical data are
available or available data do not have any time series pattern. Recently, Aviv (2001) have studied the benefits of CF techniques in a serial supply chain with a known normal demand distribution.

We also show that introducing buy-back contracts can achieve channel coordination in our system. Hitherto, most literature on supply chain coordination assume known demand distribution. Pasternack (1985) introduces buy-back contracts and Cachon and Lariviere (2000) introduces revenue-sharing contracts to achieve coordination in their systems. Both assume known demand distribution. Here we establish that coordination is also possible via buy-back contracts when demand distribution is updated periodically; however, buy-back prices have to be decided adaptively in each period. The result described above is established in Section 3. In Sections 1 and 2, we introduce and discuss three scenarios of our system, and state the important results and observations. We provide numerical examples in Section 4. Concluding comments appear in Section 5. Appendix outlines the proofs of some of the results.

1 The Models

We now formulate a two-echelon supply chain model with one supplier and two retailers. For convenience we denote the supplier by $S$ and the retailers by $R_1$ and $R_2$. The supplier sells a single product to both retailers in each period over a finite horizon of $N$ periods. The retailers operate in independent consumer markets. Let $X_{1n}^1$ and $X_{2n}^2$ denote the end consumers’ demands for the product from retailer $R_1$ and $R_2$ respectively between the decision epochs $n$ and $n+1$. We assume that in any given period $n$, $X_{1n}^1$ and $X_{2n}^2$ are independent and identically distributed (iid) with a known probability density $f(\cdot | \theta)$ and common unknown parameter (or vector of parameters) $\theta$ with realization $\theta \in \Theta$. We further assume that each retailer’s demands over different periods are also iid. Later we will indicate how we can relax some of the above assumptions. In our model, retailers are newsvendors and their demands are censored by the inventory level on hand. Therefore, they observe the sales data $x_{1n}^i$ and $x_{2n}^i$ but not the demand in period $n$. That is, $x_{in}^i = \min(X_{in}^i, y_{in}^i)$, $i = 1, 2$, where $y_{in}^i$ denotes the inventory level of retailer $R_i$ at decision epoch $n$.

In each period, the supplier buys the product from outside at a variable ordering cost of $c$ per unit and sells it to the retailers at a per-unit wholesale price $w$. Both retailers sell the product to end customers at a fixed price of $r$ per unit, get a salvage value of $h$ per unit on unsold items and incur a penalty $p$ per unit short. To avoid trivialities, it is reasonably and commonly assumed that $h < c < p$ and $h < c < w < r$. The integrated system that controls all the activities of the supplier and the retailers, gets the same salvage value as well as incurs the same shortage penalty as the retailers. We denote the integrated firm by $I$.

In our system, the retailers inform the supplier with their sales data at the end of each period. Alternatively, the supplier can see the retailers’ point-of-sale data. We adapt a Bayesian approach to learn about the demand distribution from the supplier’s perspective as well as each of the retailers perspective. To keep the framework simple, we assume that there is no leadtime for the shipment arrival both at the supplier’s site and the retail sites. The events in the system occur in the following sequence. At the beginning of each period, the supplier receives her order from outside which she ships immediately to the retailers. Retailers receive this shipment instantaneously. The customer demands occur during the period and the retailers observe the sales data. At the end of the period, the following activities are performed. The retailers convey their sales data to the supplier. The supplier as well as the retailers update their individual Bayesian scheme. Based on the updates, depending on the model in action, the retail inventory levels for the next period are decided. The supplier orders the total retail quantity from outside, and this whole process repeats at subsequent periods. The periodic sequence of events are summarized in Figure 1.
We consider two supply chain models that update the information on demand distribution differently and therefore use different inventory policies. In the first model, the supplier simultaneously update the retail demand distribution by combining information contained in both retailers’ sales data and determine appropriate shipment quantity adaptively in each period. In this system, all information become common knowledge as the supplier shares her pooled updates with the retailers. In the second model, the supplier does not use pooled updates; instead the retailers behave as isolated newsvendors. Each retailer updates his own belief about the demand based on his own sales data and determines his optimal order quantity. One of our objectives is to compare the total system profit as well as profit of each of the channel members between the above two models. We also compare the performance of the first model with that of the integrated system. We further show that the introduction of buy-back contracts (or revenue-sharing contracts) achieve channel coordination in the first model.

1.1 The Bayesian Markov Decision Process Formulation

We formulate our supply chain problems as Bayesian MDP (BMDP) models (van Hee (1978)) as follows. Let $N$ denote the finite number of decision epochs. **States:** The state space summarizes the relevant information available to each of the channel members at decision epoch $n$, $n = 1, 2, \ldots, N$. Let $M_{n} \subseteq \{f : f$ is a probability distribution on $\Theta \}$ and $M_{1} \subseteq \{all$ appropriate priors. The state space represents the set of all prior distributions when $n = 1$ and posterior distributions when $n = 2, \ldots, N$. Since we do not investigate the effect of prior specification here, we assume a fixed prior so that $M_{1} = \pi_{1}(\theta)$. Then $M_{n}$ is the set of all possible posteriors corresponding to the given prior $M_{1}(\theta)$ for the channel member $M$. Even if all of the channel members start with a common prior, it is most likely that they will eventually end up with different posterior distributions in each period. Let us denote the posterior probabilities by $M_{n}(\theta), M = I, S, R_{1}, R_{2}$, at decision epoch $n$. All of the posterior probabilities depend on both the sales observations and the retail inventory levels. The Bayesian updating mechanisms for the supplier and the integrated firm are similar. For the updating schemes of the retailers, we refer to Ding, Puterman and Bisi (2001). To keep the discussion brief, we therefore outline the updating procedure only for the integrated firm. Integrated firm’s posterior probability $\pi_{n+1}(\theta|x_{n}, x_{n}^{2})$ is given by

\[
\pi_{n+1}(\theta|x_{n}, x_{n}^{2}) = \begin{cases} 
\pi_{n+1}^{i}(\theta|x_{n}^{1}, x_{n}^{2}) & \text{if } x_{n}^{1} < y_{n}, x_{n}^{2} < y_{n} \\
\pi_{n+1}^{c}(\theta|y_{n}, x_{n}^{2}) & \text{if } x_{n}^{1} = y_{n}, x_{n}^{2} < y_{n} \\
\pi_{n+1}^{c}(\theta|x_{n}^{1}, y_{n}) & \text{if } x_{n}^{1} < y_{n}, x_{n}^{2} = y_{n} \\
\pi_{n+1}^{c}(\theta|y_{n}, y_{n}) & \text{if } x_{n}^{1} = y_{n}, x_{n}^{2} = y_{n}.
\end{cases}
\] (1)

where

\[
\pi_{n+1}(\theta|x_{n}, x_{n}^{2}) = \text{Prob}(\theta|\pi_{n}, X_{n}^{1} = x_{n}^{1}, X_{n}^{2} = x_{n}^{2}) \equiv \frac{f(x_{n}^{1}|\theta) f(x_{n}^{2}|\theta)^{\pi_{n}(\theta)}}{\int_{\Theta} f(x_{n}^{1}|\theta) f(x_{n}^{2}|\theta)^{\pi_{n}(\theta)} d\theta},
\] (2)

\[
\pi_{n+1}^{c}(\theta|y_{n}, y_{n}) = \text{Prob}(\theta|\pi_{n}, X_{n}^{1} \geq y_{n}, X_{n}^{2} \geq y_{n}) \equiv \frac{\int_{y_{n}}^{\infty} \int_{y_{n}}^{\infty} f(x_{n}^{1}|\theta) f(x_{n}^{2}|\theta)^{\pi_{n}(\theta)} dx_{n}^{1} dx_{n}^{2} \pi_{n}(\theta)}{\int_{y_{n}}^{\infty} \int_{y_{n}}^{\infty} f(x_{n}^{1}|\theta) f(x_{n}^{2}|\theta)^{\pi_{n}(\theta)} dx_{n}^{1} dx_{n}^{2} d\theta},
\] (3)

with the expressions of $\pi_{n+1}^{c}(\theta|y_{n}, x_{n}^{2})$ and $\pi_{n+1}^{c}(\theta|x_{n}^{1}, y_{n})$ are computed similarly. We use the superscripts $c_{1}, c_{2}$ and $c$ to denote demand censoring for retailer $R_{1}, R_{2}$ and both retailers respectively. Notice that we have used the same inventory level $y_{n}$ for both retailers. This is due to the
fact that we have assumed identical demand distributions for the retailers in our system. For more
detail discussion on (2)-(3), we refer to Ding, Puterman and Bisi (2001).
Correspondingly, the Bayesian estimate of the marginal demand distribution process \( \{I m_n'(x), n = 1, \ldots, N\} \) satisfies
\[
I m_n'(x) = \int_0^\infty I m_n'(x, x^2)dx^2 = \int_0^\infty I m_n'(x^1, x)dx^1,
\]
where
\[
I m_n'(x^1, x^2) = \int_\Theta (f(x^1|\theta)f(x^2|\theta)\pi_n'(\theta|x_{n-1}, x_{n-1}^2)d\theta.
\]
Since \( I \pi_n' \) is calculated differently with exact demand observation(s) than with censored observa-
tion(s) as indicated in (1), the marginal demand distribution \( I m_n' \) can be distinguished into the
corresponding four cases as well, i.e.,
\[
I m_n'(x) = \begin{cases} 
I m_n(x) & \text{if } x_{n-1}^1 < y_{n-1}, x_{n-1}^2 < y_{n-1} \\
I m_n^c(x) & \text{if } x_{n-1}^1 = y_{n-1}, x_{n-1}^2 < y_{n-1} \\
I m_n^a(x) & \text{if } x_{n-1}^1 < y_{n-1}, x_{n-1}^2 = y_{n-1} \\
I m_n^a(x) & \text{if } x_{n-1}^1 = y_{n-1}, x_{n-1}^2 = y_{n-1}.
\end{cases}
\]
Similar to (1), (4) and (5), we can obtain the supplier’s Bayesian updates \( S \pi_{n+1}'(x|x_n^1, x_n^2), S m_n'(x^1, x^2) \)
and \( S m_n'(x) \). For the retailer’s individual updates \( R_i \pi_{n+1}'(\theta|x_n) \) and \( R_i m_n'(x), i = 1, 2, \) we refer to
Ding, Puterman and Bisi (2001).
\textbf{Actions:} Since any inventory level can be chosen at any decision epoch, the action set \( A_s = [0, \infty) \)
for each \( s \in AS_n, M = I, S, R_1, R_2. \)
\textbf{Transition Probabilities:} At decision epoch \( n \), the transition probability (density) for the inte-
rated firm’s BMDP may be specified for any \( y_n \in [0, \infty) \) as
\[
p(I \pi_{n+1}'|I \pi_n', y_n) = \begin{cases}
I m_n'(x_n^1, x_n^2) & \text{if } x_n^1 < y_n, x_n^2 < y_n \\
\int_{y_n}^\infty I m_n'(x_n^1, x_n^2)dx_2 & \text{if } x_n^1 = y_n, x_n^2 < y_n \\
\int_{y_n}^\infty I m_n'(x_n^1, x_n^2)dx_2 & \text{if } x_n^1 < y_n, x_n^2 = y_n \\
\int_{y_n}^\infty \int_{y_n}^\infty I m_n'(x_n^1, x_n^2)dx_1dx_2 & \text{if } x_n^1 = y_n, x_n^2 = y_n.
\end{cases}
\]
The transition probability equals 0 for posteriors that cannot be attained from \( I \pi_n' \). Note that
the transition probabilities depend on actions. Similarly, we can write the supplier’s transition
probability \( p(S \pi_{n+1}'|S \pi_n', y_n) \). For the retailer’s transition probabilities, we refer to Ding, Puterman
and Bisi (2001).

\section{Analysis of the Models}

In this section we will establish important structural properties of the optimal inventory policies
for the supply chain models discussed in Section 1. The retailers are Bayesian newsvendors with
unobservable lost sales and continuous non-negative demand. Unobservable lost sales complicates
the problems greatly. Because the transition probabilities depend on the actions, the demand
distribution updating depends on the entire history. This means that analytic solutions are not
readily available and would be very hard to obtain. We therefore attempt to establish structural
results and carry out some numerical calculations. To this end, we first introduce some notation.
For \( n = 1, 2, \ldots, N \), let
\( I^\prime M''(x) = \int_0^x I^\prime M''(s) ds, \quad S^\prime M'(x) = \int_0^x S^\prime M'(s) ds, \quad R^\prime M'(x) = \int_0^x R^\prime M'(s) ds, \quad i = 1, 2. \)

\( I^\prime y_n^*: \) optimal policy for each retailer in period \( n \) with state \( I^\prime \pi_n'(\theta|x_{n-1}^1, x_{n-1}^2) \).

\( I^\prime y_n^{BN} = [I^\prime M''(\frac{x_{n-1}^1 + x_{n-1}^2}{2})]^{-1} : \) Bayesian newsvendor (myopic) policy for each retailer in period \( n \) with state \( I^\prime \pi_n'(\theta|x_{n-1}^1, x_{n-1}^2) \).

Notice that each of \( I^\prime y_n^* \) and \( I^\prime y_n^{BN} \) consists of four cases corresponding to the four cases of the exact demand observation(s) and censored observation(s) for two retailers as indicated by (1) and (6).

For our analysis we will need the concept of likelihood ratio ordering whose definition is provided below from Ross (1983).

**Likelihood Ratio Ordering** Let \( X \) and \( Y \) denote continuous non-negative random variables having densities \( f \) and \( g \) respectively. We say that \( X \) is larger than \( Y \) in the likelihood ratio sense, and write \( X \geq_{LR} Y \) if

\[
\frac{f(x)}{g(x)} \leq \frac{f(y)}{g(y)} \quad \text{for all } x \leq y.
\]

That is \( X \geq_{LR} Y \) if the ratio of their respective densities, \( f(x)/g(x) \), is nondecreasing in \( x \).

### 2.1 The Integrated System

We first analyze the integrated system.

**Expected Profits:** The Bayesian expected profit with prior distribution \( I^\prime \pi_n^* \) and each retailer’s inventory level \( y_n \) is given by

\[
I^\prime R_B(I^\prime \pi_n', y_n) = 2 \left[-cy_n + r \int_0^{y_n} x^t m_n'(x) dx + ry_n \int_{y_n}^{\infty} m_n'(x) dx \right.
\]

\[
+ h \int_0^{y_n} (y_n - x) I^\prime m_n'(x) dx - p \int_{y_n}^{\infty} (x - y_n) I^\prime m_n'(x) dx \right]
\]

\[
= 2 \left[(r - c)y_n - (r - h) \int_0^{y_n} \left( \int_0^x I^\prime m_n'(s) ds \right) dx - p \int_{y_n}^{\infty} (x - y_n) I^\prime m_n'(x) dx \right],
\]

where the last equation is obtained via integration by parts. We assume that \( I^\prime R_B(I^\prime \pi_{N+1}^*, y_{N+1}) = 0. \) Note that \( I^\prime R_B(I^\prime \pi_n', y_n) \) is continuous and concave in \( y_n \).

**Optimality Equations:** Under the total expected reward criterion, the optimality equations for the integrated firm are given by

\[
 u_n(I^\prime \pi_n') = \max_{y_n \in R_+} \left\{ I^\prime R_B(I^\prime \pi_n', y_n) + \int_0^{y_n} \int_0^{y_n} u_{n+1}(I^\prime \pi_{n+1}^*(|x_1^1, x_2^1, x_1^2, x_2^2)) I^\prime m_n'(x_1^1, x_2^1) dx_1^1 dx_2^1 \right.
\]

\[
+ \int_0^{y_n} u_{n+1}(I^\prime \pi_{n+1}^*(|y_n, x_1^2)) \int_{y_n}^{\infty} I^\prime m_n'(x_1^1, x_2^1) dx_1^1 \right.
\]

\[
+ \int_0^{y_n} u_{n+1}(I^\prime \pi_{n+1}^*(|x_1^1, y_n)) \int_{y_n}^{\infty} I^\prime m_n'(x_1^1, x_2^1) dx_2^1 \right.
\]

\[
+ u_{n+1}(I^\prime \pi_{n+1}^*(|y_n, y_n)) \int_{y_n}^{\infty} \int_{y_n}^{\infty} I^\prime m_n'(x_1^1, x_2^1) dx_1^1 dx_2^1 \right\}
\]

for \( n = 1, \ldots, N \), with the boundary condition

\[
u_{N+1}(I^\prime \pi_{N+1}^*) = 0
\]

for all \( I^\prime \pi_{N+1}^* \). The second term on the right hand side of (7) is the expected profit at \( n + 1 \) if both \( X_n^1 \) and \( X_n^2 \) are fully observed (i.e. \( X_n^1, X_n^2 < y_n \)) and the fifth term is the expected profit if both
Theorem 1. Suppose that the retailers’ demands at decision epoch \( n \), \( X^i_n \), \( n = 1, 2, \ldots, N \), \( i = 1, 2 \), are non-negative iid random variables with density \( f(x|\theta) \), and suppose \( f(x|\theta) \) is likelihood ratio increasing in \( \theta \). Then the optimal order quantity \( I^*_n \) at period \( n \), \( n = 1, 2, \ldots, N-1 \), of an \( N \)-period problem with unobservable lost sales satisfies

\[
(c-h)^{I}M'_{n}(y_n) = \left[1 - I^*M_{n}(y_n)\right] \left[r + p + I^*_n(y_n) - c\right],
\]

where

\[
I^*_n(y_n) = \frac{1}{1 - I^*M_{n}(y_n)} \frac{dI_n(y_n)}{dy_n}
\]

with \( I^*M_{n}(y_n) \) being the last four terms inside the bracket on the right hand side of (7).

Theorem 2. Suppose the hypotheses of Theorem 1 hold. Then the optimal inventory level of each retailer satisfies

\[
I^*_n \geq I^*_{BN}, \quad n = 1, 2, \ldots, N-1, \quad \text{and} \quad I^*_{N} = I^*_{BN}.
\]

The intuition behind the above result can be given in the same way as by Ding, Puterman and Bisi (2001). Sitting at each decision epoch \( 1, 2, \ldots, N-1 \), the manager of the integrated firm might want the retailers to sacrifice the profit maximization only for the current one-period (myopic) problem so that the system can obtain extra information about the demand distribution in order to make better decisions at subsequent epochs.

Let us denote the total expected profit of the integrated system by \( T \) which sets a performance benchmark. We compare this profit with the total channel profits of the following two systems.

2.2 System A: Supplier Shares Pooled Demand Information

In this system, the supplier sells the product to the retailers at a wholesale price \( w \) per unit, \( c < w < r \). Apart from this, the basic functional structure of this system is similar to the integrated system. The supplier’s updating mechanism of the marginal demand distributions is similar to that of the integrated system and she conveys her pooled updates to both retailers. Thus, all information are common knowledge in this system also. Based on the updates \( S\pi_n'(\theta) \), \( S^1m'_n(x, x^2) \) and \( S^2m'_n(x) \), the supplier determines a common shipment quantity \( S^*y^*_n \) that optimize the retailers’ combined profit over the entire horizon. Therefore, the optimality equations for this system are simply obtained from (7) by replacing \( c \) by \( w \), \( I^*_n \) by \( S\pi_n'(\theta) \), \( I^*m^*_n(x^1, x^2) \) by \( S^1m'_n(x^1, x^2) \), and \( I^*m^*_n(x) \) by \( S^2m'_n(x) \). It is easy to see that \( S^*y^*_n < I^*_n \), a consequence of ‘double marginalization’. This system...
can be viewed as a special type of Vendor Managed Inventory (VMI) system where the retailers share their point-of-sale data with the supplier and in turn the supplier determines the retail inventory levels by utilizing the pooled information on demand. The model is of collaborative forecasting and replenishment (CFR) type, via dynamic Bayesian updating of the demand distribution.

The supplier’s total profit over the entire horizon is \( S := 2 \sum_{n=1}^{N} (w - c) y_{n}^* \). Let us denote each retailer’s total expected profit by \( P \). Thus, the total expected profit for the whole channel over the entire horizon equals \( S + 2P \).

2.3 System B: Supplier Does Not Use Pooled Demand Information

In this system, the retailers maintain their individual updates \( R_{i} \pi_{n}^{\prime}(\theta) \) and \( R_{i} m_{n}^{\prime}(x) \), \( i = 1, 2 \), separately. They compute their optimal policies based on these updates.

**Retailer’s Expected Profits:** The Bayesian expected profit with respect to \( R_{i} \pi_{n}^{\prime} \) is

\[
R_{i} R_{B}(R_{i} \pi_{n}^{\prime}, y_{n}) = \left[(r - w) y_{n} - (r - h) \int_{0}^{y_{n}} \left(\int_{0}^{y} R_{i} m_{n}^{\prime}(s) ds\right) dx \right. \\
- p \int_{y_{n}}^{\infty} (x - y_{n}) R_{i} m_{n}^{\prime}(x) dx, \quad i = 1, 2,
\]

**Retailer’s Optimality Equations:** The optimality equations for the retailers are

\[
u_{n}(R_{i} \pi_{n}^{\prime}) = \max_{y_{n} \in R_{i}} \left\{ R_{i} R_{B}(R_{i} \pi_{n}^{\prime}, y_{n}) + \int_{0}^{y_{n}} u_{n+1}(R_{i} \pi_{n+1}(\cdot|y_{n}))[1 - R_{i} M_{n}^{\prime}(y_{n})] \right\}, \quad i = 1, 2, \tag{9}
\]

for \( n = 1, \ldots, N \), with the boundary condition

\[
u_{N+1}(R_{i} \pi_{N+1}^{\prime}) = 0.
\]

To state a result below, we introduce some notation. Let

\[
R_{i} y_{n}^{*} : \text{optimal policy for retailer } R_{i} \text{ in period } n \text{ with state } R_{i} \pi_{n}^{\prime}(\theta|x_{n-1}), i = 1, 2.
\]

\[
R_{i} y_{n}^{BN} = [R_{i} M_{n}^{\prime}(\frac{r+h-w}{r})]^{-1} : \text{Bayesian newsvendor (myopic) policy for retailer } R_{i} \text{ in period } n \text{ with state } R_{i} \pi_{n}^{\prime}(\theta|x_{n-1}).
\]

Then mimicking the methodology developed in Ding, Puterman and Bisi (2001), we will obtain the following result.

**Theorem 3.** Under the hypotheses of Theorem 1, the optimal order quantity \( R_{i} y_{n}^{*} \) of retailer \( R_{i}, i = 1, 2 \), at period \( n, n = 1, 2, \ldots, N - 1 \), satisfies

\[(w - h)R_{i} M_{n}^{\prime}(y_{n}) = \left[1 - R_{i} M_{n}^{\prime}(y_{n})\right] \left[r + p - \frac{du_{n+1}(R_{i} \pi_{n+1}^{\prime}(y_{n}))}{dy_{n}}\right]
\]

\[
\left. + \left(u_{n+1}(R_{i} \pi_{n+1}^{\prime}(y_{n}))- u_{n+1}(R_{i} \pi_{n+1}^{\prime}(y_{n}))\right) \frac{R_{i} m_{n}^{\prime}(y_{n})}{1 - R_{i} M_{n}^{\prime}(y_{n})} - w\right]. \tag{10}
\]

Moreover,

\[
R_{i} y_{n}^{*} \geq R_{i} y_{n}^{BN}, \quad n = 1, 2, \ldots, N - 1, \quad \text{and} \quad R_{i} y_{N}^{*} = R_{i} y_{N}^{BN}.
\]

With the retailers’ policies \( R_{i} y_{n}^{*} \), \( n = 1, \ldots, N, i = 1, 2 \), the supplier’s total profit over the entire horizon is \( S' := \sum_{n=1}^{N} \sum_{i=1}^{2} (w - c) R_{i} y_{n}^{*} \). Let us denote the total expected profit of retailer \( R_{i} \) by \( P_{i}', i = 1, 2 \). Therefore, the total expected profit for the entire channel equals \( S' + P_{1}' + P_{2}' \).
2.4 Some Observations

Comparing the three models described above, we now make the following observations:

(a) \( S' + P'_1 + P'_2 \leq S + 2P < T; \)

(b) \( P'_1 + P'_2 \leq 2P. \)

Though the above observations are easy to understand intuitively, it will be hard to prove these analytically due to the technical intricacies inherent in the Bayesian formulation of the models. The above observations tell us that by using the pooled information obtained from the retailers’ sales data, one would be able to increase the total channel profit as well as the combined profit of the retailers. However, whether \( S' \leq S, \) or both of \( P'_1 \) and \( P'_2 \leq P \) are true or not would depend on the cost and revenue parameters as well as the number of periods of the specific problem we address. It will be interesting to construct some example systems where the use of the pooled demand information would benefit all the channel members, that is, \( S' \leq S, \) and \( P'_1, P'_2 \leq P \) hold. Moreover, to get an idea on the amount of gain from the pooling of demand information, we should run simulations and estimate the gaps in the total channel profits for the three systems as described by the inequalities in (a) above.

2.5 Example: Gamma-Weibull Model

The Weibull Distribution belongs to the family of newsboy distributions characterized by Braden and Freimer (1991). It has a fixed dimensional sufficient statistic under exact and right-censored observations. Let \( F(x), x > 0, \) be the CDF for demand defined as

\[
F(x) = 1 - e^{-\theta x^b}
\]

The parameter \( \theta \) is unknown. The sufficient statistic for \( \theta \) after \( n \) sales observations is \( [\alpha; \sum_{i=1}^{n} x_i^b + \sum_{i=1}^{n} y_i^b], \) where \( \alpha \) is the number of exact observations and \( (x_1, \ldots, x_n) \) is the sequence of exact observations. Suppose the prior distribution for \( \theta \) is Gamma with parameters \( \alpha_0 \geq 0 \) and \( \beta_0 > 0. \) Define the prior PDF for \( \theta \) as

\[
\pi(\theta|\alpha_0, \beta_0) = \frac{\beta_0^{\alpha_0} \theta^{\alpha_0-1} e^{-\theta \beta_0}}{\Gamma(\alpha_0)}
\]

The the sufficient statistic in period \( n \) is defined recursively as \( \alpha_n = \alpha_{n-1} + I_{[x_n<y_n]} \) and \( \beta_n = \beta_{n-1} + x_n^b I_{[x_n<y_n]} + y_n^b I_{[x_n\geq y_n]}, n > 0. \) The predictive CDF for \( X_n \) given \( \alpha_n \) and \( \beta_n \) is

\[
M'_n(x_n) = 1 - \left[ \frac{\beta_n}{\beta_n + x_n^b} \right]^{\alpha_n}
\]

The derivation of \( M'_n(x) \) is in Braden and Freimer (1991). In the case where sales is observed from two retailers, \( \alpha_n = \alpha_{n-1} + I_{[x_1^c<y^c_1]} + I_{[x_2^c<y^c_2]} \) and \( \beta_n = \beta_{n-1} + \sum_{i=1}^{2} \left( x_n^b I_{[x_n^c<y_n^c]} + y_n^b I_{[x_n^c\geq y_n^c]} \right), n > 0. \)

Let \( b = 5, \alpha_0 = 1, \) and \( \beta_0 = 4.7889 \times 10^8. \) The predictive PDF for \( X_n \) is given in Figure 1. Suppose \( N = 3, r = 2, c = 1, p = 1.25, \) and \( h = .25. \) The stochastic dynamic program was implemented using a 10th degree polynomial to approximate the cost-to-go as a function of \( \beta_n \) for each possible value of \( \alpha \) for \( n = 2, 3. \) The grid of \( \beta_n \) values used was the sequence \( \{10^{ak}\}, \) \( k = 1, \ldots, 61, a_1 = 3, a_k = a_{k-1} + .1. \) The root mean squared error of the approximation was less than \( 1 \times 10^{-7} \) in period 3. Figure 2 shows the integrated channel maximum expected profit in
Figure 1: Predictive PDF for the Gamma-Weibull example.

Figure 2: Maximum expected profit for the integrated channel in period 3 as a function of the sufficient statistic.
period 3 for the grid of \( \beta_3 \) values when \( \alpha_n = 1 \) and \( \alpha_3 = 5 \). The maximum total expected profit for the integrated channel at \( n = 1 \) given \( \alpha_0 \) and \( \beta_0 \) is approximately \( T = 240.5 \).

Now suppose \( w = 1.5 \) and the retailers have global sales data. Then \( P \approx 18.7 \) and \( S \approx 158.6 \). In contrast, \( P' \approx 16.5 \) and \( S' \approx 66.9 \) when the retailers are limited to local sales data.

## 3 Channel Coordination: Buy-Back Contracts

In this section we examine the buy-back contracts that achieve channel coordination for the System A. Under such a contract, the supplier announces a fixed wholesale price \( w \) in the beginning of the first period and agrees to purchase all unsold goods from the retailers at the end of each period for buy-back price \( b \) per unit, where \( h < b < w \). We will see that in order to achieve channel coordination, this \( b \) has to be decided adaptively in each period. The supplier earns a fixed salvage value of \( h \) per unit on all returned items.

Rewriting (8) in Theorem 1 we see that \( I^*_n \) satisfies

\[
I'M'_n(y_n) = \frac{r + p + I_n(y_n) - c}{r + p + I_n(y_n) - h},
\]

(14)

On the other hand, with the buy-back contracts in effect, similar to Theorem 1, the optimal shipment quantity \( S^*_n \) to each retailer is obtained from

\[
S'M'_n(y_n) = \frac{r + p + S_i(y_n) - w}{r + p + S_i(y_n) - b},
\]

(15)

where the expression for \( S_i(y_n) \) is similar to that of \( I_i(y_n) \), only replace \( I_i(y_n) \) by \( S_i(y_n) \) and \( M_i(x^1, x^2) \) by \( S_i(x^1, x^2) \).

If the supplier chooses buy-back prices that coordinate the channel for the System A, then in any period \( n \), we will have \( I^*_n = I^*_n \), which consequently would imply that \( S_n(y_n) \) and \( S_i(y_n) \) coincide with \( I_n'(y_n) \) and \( I_i(y_n) \) respectively. Therefore, equating the right hand sides of (14) and (15) would lead to

\[
b \equiv b_n(w) = \frac{r + p + I_n(y_n) - h}{r + p + I_n(y_n) - c} (w - c) + h.
\]

(16)

Since \( I_i(y_n) \) can be estimated from (14) at \( y_n = I^*_n \), with \( I^*_n \) being enumerated by the backward induction of the optimality equations (7), we can compute \( b_n(w) \) from (16). Notice that, the supplier must choose the buy-back price adaptively in each period in order to achieve channel coordination.

With coordinating buy-back contracts in action, the supplier has the flexibility of choosing a wholesale price that maximizes her profit

\[
\Pi^S(w; b_n(w), I^*_n, n = 1, \ldots, N) = \sum_{n=1}^{N} [2(w - c)I^*_n - 2(b_n(w) - h) \int_0^{I^*_n} (I^*_n - x)M_n'(x)dx]
\]

(17)

Notice that from (15) we have

\[
\frac{db_n(w)}{dw} = \frac{1}{I_n'(I^*_n)}.
\]
Now differentiating (17) with respect to \( w \) we get

\[
\frac{d\Pi_s}{dw}(w, b_n(w), I^*_n, n = 1, \ldots, N) = 2 \sum_{n=1}^{N} \left[ I^*_n - \frac{1}{M'_n(I^*_n)} \left( I^*_n M'_n(I^*_n) - \int_0^{I^*_n} x' m'_n(x) dx \right) \right]
\]

\[= 2 \sum_{n=1}^{N} \frac{1}{M'_n(I^*_n)} \int_0^{I^*_n} x' m'_n(x) dx > 0 \tag{18}\]

In summary, we have the following result.

**Proposition 1.**

(i) For any \( w \), there is coordinating buy-back contract \( b_n(w), n = 1, \ldots, N; \)

(ii) Supplier’s profit function \( \Pi_s(w, b_n(w), I^*_n, n = 1, \ldots, N) \) is an increasing function of \( w \).

In view of the above proposition, we observe that it is possible to choose a wholesale price \( w \) (subject to the retailers’ approval) and coordinating buy-back prices \( b_n(w), n = 1, \ldots, N; \) (subject to the supplier’s approval) that will support an arbitrary division of the profit between the supplier and the retailers.

**Remark 1.** In our discussion so far we have assumed that in any period \( n \), the retail demands \( X^1_n \) and \( X^2_n \) are iid. We may relax the condition on identical distribution in the following way. Observe that our methodology can be easily extended to the cases where \( X^1_n \) and \( X^2_n \) are such that \( X^1_n \) and \( (c_1 + c_2 X^2_n) \) have the same probability distribution for some constants \( c_1 \) and \( c_2 \). This is what happens in practice when the retailers sell either the same product at different locations or similar but distinct and nonsubstitutable products at the same location (see Lariviere and Porteus (1999)).

**Remark 2.** We can easily extend the work to include any number of retailers. However, this will only complicate the model without helping to gain any extra insight.

### 4 Conclusion

For a supply chain with one supplier and multiple independent retailers, we have established that when the demand distribution is unknown and lost sales are unobservable, significant benefits can be observed if the information contained in all of the retailers sales data are properly pooled and utilized to determine the retail inventory levels. While we have concentrated on a finite-horizon problem with newsvendor retailers, we are confident that similar observations can be made in more complex systems.

We view this paper as a first and general step to the analysis of demand censoring effect on supply chain inventory decisions. Several important questions remain which we describe below:

i) Extension of the methodologies developed in this paper to unknown and non-stationary demand distributions;

ii) Study of the Bayesian collaborative forecasting (CF) techniques when demands are dependent over periods;

iii) Extension of the results to price-dependent demands;
iv) Investigation of Bayesian techniques when the retailers’ demands are correlated;

v) How to modify the Bayesian approach if there is a lead time for the supplier’s order?

vi) Extension to base stock and $(s, S)$ inventory systems with uncertain and censored demand;

**Appendix**

**Proof of Theorem 1.** The ideas and methodology of the proof closely follow the proof of Theorem A in Ding, Puterman and Bisi (2001). We therefore provide here only a brief outline of the proof, omitting the details.

Let

\[ J(l \pi_n', y_n) \]

\[ = IRI_B(l \pi_n', y_n) + \int_0^{y_n} \int_0^{y_n} u_{n+1}(l \pi_{n+1}(\cdot|y^1, x^2))' m_n'(x^1, x^2) dx^1 dx^2 \]

\[ + \int_0^{y_n} u_{n+1}(l \pi_{n+1}^c(\cdot|y_n, x^2)) \left( \int_0^{\infty} I'm_n'(x^1, x^2) dx^1 \right) dx^2 \]

\[ + \int_0^{y_n} u_{n+1}(l \pi_{n+1}^c(\cdot|y_n, x^1)) \left( \int_0^{\infty} I'm_n'(x^1, x^2) dx^2 \right) dx^1 \]

\[ + u_{n+1}(l \pi_{n+1}^c(\cdot|y_n, y_n)) \int_0^{y_n} \int_0^{y_n} I'm_n'(x^1, x^2) dx^1 dx^2 \]

\[ \equiv IRI_B(l \pi_n', y_n) + I_I(y_n). \]  

(19)

Then the optimal policy is obtained from

\[ I_y^* \in \arg \max_{y_n} J(l \pi_n', y_n), \text{for } n = 1, 2, \ldots, N. \]

We can argue as by Ding, Puterman and Bisi (2001) to verify that for continuous demand, \( J(l \pi_n', y_n) \) is differentiable in \( y_n \). Also, notice from the expression of \( I_I(y_n) \) (refer to (19)) that the first and the fourth terms are symmetric with respect to the variables \( x^1 \) and \( x^2 \), whereas the second and the third terms are essentially equal. Therefore, by Leibnitz’s rule we get the following

\[ \frac{dJ(l \pi_n', y_n)}{dy_n} \]

\[ = \frac{d}{dy_n} IRI_B(l \pi_n', y_n) + \frac{d}{dy_n} I_I(y_n) \]

\[ = (r + p - c) - (r + p - h)I'M_n'(y_n) + 2 \int_0^{y_n} u_{n+1}(l \pi_{n+1}(\cdot|y^1, x^2))' m_n'(x^1, y_n) dx^1 \]

\[ + 2 \int_0^{y_n} \int_0^{\infty} u_{n+1}(l \pi_{n+1}^c(\cdot|y^1, y_n)) \left( \int_0^{\infty} I'm_n'(x^1, x^2) dx^2 \right) dx^1 \]

\[ + 2u_{n+1}(l \pi_{n+1}^c(\cdot|y_n, y_n)) \int_0^{y_n} I'm_n'(x^1, y_n) dx^1 - 2 \int_0^{y_n} u_{n+1}(l \pi_{n+1}^c(\cdot|y^1, y_n))' m_n'(x^1, y_n) dx^1 \]

\[ + \frac{du_{n+1}(l \pi_{n+1}^c(\cdot|y_n, y_n))}{dy_n} \int_0^{y_n} \int_0^{\infty} I'm_n'(x^1, x^2) dx^1 dx^2 \]

\[ - 2u_{n+1}(l \pi_{n+1}^c(\cdot|y_n, y_n)) \int_0^{y_n} I'm_n'(x^1, y_n) dx^1. \]  

(20)

Now, since \( J(l \pi_n', y_n) \) (defined in (19)) is a concave function, we can obtain the optimal order quantity by solving for \( y_n \) from \( \frac{dJ(l \pi_n', y_n)}{dy_n} = 0 \). Setting \( \frac{d}{dy_n} l \pi_n', y_n = 0 \) and rearranging terms gives (8).
Proof of Theorem 2. To show \( I_n^* \geq I_{y_{BN}}^n \) for any \( n = 1, 2, \ldots, N - 1 \), it suffices to show that \( \frac{dI_n^*(y_n)}{dy_n} \bigg|_{y_n = I_{y_{BN}}^n} \geq 0 \) where the expression of \( \frac{dI_n^*(y_n)}{dy_n} \) is as given in (20). Since \( \frac{dI_n}{dy_n} \bigg|_{y_n = I_{y_{BN}}^n} = 0 \), we only need to show that at \( y_n = I_{y_{BN}}^n \), \( \frac{dI_n}{dy_n} \geq 0 \).

Notice that we can rearrange the terms in \( \frac{dI_n}{dy_n} \) (refer to (20)) to separate into two parts in the following way

\[
\frac{dI_n}{dy_n} = \left[ 2 \int_0^{y_n} \left( \frac{du_{n+1}(I_{\pi_n+1}(\cdot|x^1, y_n))}{dy_n} \left( \int_{y_n}^{\infty} I_{m_n}(x^1, x^2)dx^2 \right) - \left( u_{n+1}(I_{\pi_n+1}(\cdot|x^1, y_n)) - u_{n+1}(I_{\pi_n+1}(\cdot|x^1, y_n)) \right) \right) \right] \]

\[
\left. \frac{du_{n+1}(I_{\pi_n+1}(\cdot|y_n, y_n))}{dy_n} \int_{y_n}^{\infty} I_{m_n}(x^1, x^2)dx^2 \right) \left. \int_{y_n}^{\infty} I_{m_n}(x^1, x^2)dx^2 \right) \]

\[
- 2 \left( u_{n+1}(I_{\pi_n+1}(\cdot|y_n, y_n)) - u_{n+1}(I_{\pi_n+1}(\cdot|y_n, y_n)) \right) \int_{y_n}^{\infty} I_{m_n}(x^1, x^2)dx^2 \right) \]

(21)

Next, following the technique developed by Ding, Puterman and Bisi (2001) and keeping in mind that here we wish to maximize the profit instead of minimizing the cost, we can show that at any value of \( y_n \), both of the bracketed parts in (21) are nonnegative, where the non negativity of the first part is a consequence of the fact that its integrand is nonnegative. This completes the proof of the theorem.

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