Discrete Prices and Competition in a Dealer Market

Job Market Paper[1]

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Abstract

The goal of this paper is to investigate the relation between price discreteness and competition in a dealer market, and to study their impact on the welfare of different market participants (dealers and investors). We present a model featuring a finite number of dealers competing in prices for supplying liquidity to a forthcoming market order. Our main result is that there exists an interaction between price discreteness and competition. When competition is lax, dealers prefer fine price grids (small tick size), while when competition is intense they prefer coarse price grids (large tick size). When competition is lax, investors' attitude towards the tick size is ambiguous with some preference towards larger ticks. When competition among dealers is intense, investors prefer fine price grids. The rate of convergence of markets towards perfect competition is slower, the finer is the grid of prices. This means that the smaller is the tick size, the larger is the number of dealers required to assure a "close to competitive" outcome. Our results regarding low levels of competition among dealers stand in contrast to the standard views and common wisdoms, suggesting that refining the grid of prices is always beneficial for investors. Our analysis suggests that when deciding on the "correct" tick size for a specific stock, market designers should take into account the level of competition among dealers for this stock. While this paper refers to dealer markets only, we claim that our main results and intuitions are applicable also to other market mechanisms in which liquidity suppliers compete using limit orders for anticipated market orders.

1 Introduction

Prices in everyday life are discrete. We cannot pay in cash amounts smaller than one penny since our coins do not allow it, however price discreteness in various markets exceeds this physical limitation by far. For example, e-Bay - the largest auctions site on the Web, requires that bids are in multiples of approximately 1%-2% of reserve prices. Thus, cars are sold in multiples of about $100, cameras in multiples of $5-$10, and so on. Security markets are no exception. Until recently, US stock markets traded stocks in multiples of $\$\frac{1}{16}$, and previously $\$\frac{1}{8}$. As of April 2001 all US stock markets are using a decimal system, and most prices are quoted in multiples of one cent.

The choice of the minimum price variation (hereafter called "the tick size") in any trading mechanism is at the hands of the mechanism designer. It is commonly believed that this decision can alter the division of gains from trade between different market participants. Moreover, decreasing the tick size is considered welfare improving, in the sense that total gains from trade in the market increase. This is so, because a positive tick size is considered to be a "friction", that prohibits trade from taking place in occasions when there are positive potential gains from trade.

Furthermore, changing the tick size has been used in financial markets as a policy tool to transfer rents from one type of market participant to another. For example, it is commonly believed that a large tick size benefits professional traders (e.g. dealers who quote prices). Thus, decreasing the tick size in markets such as NASDAQ was aimed to deprive dealers from some of their profits, and transfer these profits to investors.

The goal of this paper is to investigate the welfare consequences of imposing a positive tick size on a dealer market, and relate these consequences to the level of competition among dealers. We present a model featuring a finite number of dealers for a specific stock. The dealers compete for supplying liquidity to a forthcoming market order by submitting simultaneous quotes. The market order may or may not be submitted, depending on the reservation value of a representative investor. Each dealer has his own private valuation of the stock, driven by his idiosyncratic inventory position. Dealers' quotes are restricted to a finite grid of prices, even though their private valuations may take any value in a finite interval.

We model this interaction as a game of incomplete information (Harsanyi 1967-1968), and study the welfare properties of non-decreasing, symmetric, pure strategy Nash equilibria. We view this setting as one out of many similar encounters between dealers and investors. Thus, when deciding on the "correct" tick size, the market designer would like to maximize ex-ante welfare, which is the expected gains from trade calculated before any market participant knows his reservation value. We use this ex-ante approach to assess the welfare properties of different discretization policies.

Our main result is that there exists an interaction between the level of price discreteness in the market and the level of competition (number of dealers). Specifically, when the level of competition among dealers in high, they prefer a coarse grid of prices (large tick size), whereas when the level of competition is low, they prefer a fine grid of prices (small tick size). As for investors, if competition among dealers is intense, they prefer a small tick size. If, on the other hand, the competition is lax, there is ambiguity in their preference, with some tendency towards larger ticks. Moreover, we show that the rate of convergence from the monopolistic dealer case to the perfectly competitive case (infinite number of dealers) is slower, the finer is the grid of prices. This means that the smaller is the tick size, the larger is the number of dealers required to assure an approximately perfectly competitive outcome.

The interaction between discrete prices and competition may be drawn from basic intuitions regarding competition in oligopolistic markets. In order to articulate this intuition we refer to the two extreme cases: the monopolistic dealer case, and the perfectly competitive case. Then, we study the gradual movement from the former case to the latter.

Consider first the case of a monopolistic dealer. As any monopoly, this dealer would like to maximize his ex-ante profits, by submitting the optimal quote given the distribution of reservation values of the upcoming market order. In a market with a continuous grid of prices the dealer is able to submit such an optimal quote. However, a positive tick size prevents him from doing so, and his expected profit is reduced. Thus, a positive tick size serves as a restraining device, preventing the monopolistic dealer from fully exploiting his market power. By increasing the tick size, investors enjoy the extra welfare extracted from the monopolistic dealer, however, a too large tick size might decrease the probability of transaction. These two contradicting forces make the attitude of investors to the tick size in this case ambiguous. This attitude depends on the distribution of reservation values.

Consider now the perfectly competitive case - an infinite number of dealers. In this case, dealers are forced to submit competitive quotes. This means that their quotes are as close as possible to their reservation values, given that quotes must lie on a discrete grid. No dealer shades his quote by more than one tick. As a result, in the perfectly competitive case, the tick size is the only source of profits to dealers, and it may be viewed as a coordination device that enables them to earn positive profits despite the severe competition. Thus, in the perfectly competitive case, decreasing the tick size, decreases the welfare of dealers. The welfare loss of dealers is transferred to the public of investors who enjoy an increase in welfare following a decrease in tick size. Total market welfare increases as well, reflecting the fact that a decrease in tick size increases the ex-ante probability of transaction.

Intermediate levels of competition feature gradual movement from the monopolistic case to the perfectly competitive case. Thus, when the number of dealers is small, they prefer small ticks, since a large tick size prevents them from exploiting their full scale of market power. On the other hand, when the number of dealers is large, dealers prefer large ticks. As for the investors: when the level of competition is high, they prefer fine price grids, while their relation to the tick size in the case of lax competition among dealers is ambiguous. On the one hand, a large tick size in this case restrains dealers and transfers gains to the investors, but on the other hand it might decrease the probability of transaction and hurt the investors. The result regarding the rate of convergence to the perfectly competitive case reinforces the interaction between price discreteness and competition.

While this paper refers to dealer markets only, we claim that our main results and intuitions are applicable also to other market mechanisms in which liquidity suppliers compete using limit orders for anticipated market orders. The role of dealers in our model is played by liquidity suppliers and the role of investors is played by liquidity demanders. The important feature needed in order to facilitate our analysis is that liquidity suppliers act before the liquidity demanders by submitting limit orders. This "first mover advantage" given to liquidity suppliers is sufficient to create the interaction between price discreteness and competition.

Our results regarding low levels of competition stand in contrast to the standard views and common wisdoms that led to the decimalization process. These views suggest that decreasing the tick size is *always* beneficial for the public of investors. Our results suggest that the level of competition among dealers for a specific security should be taken into account when deciding on the tick size. In particular, if the number of dealers is low, setting a larger tick size might hurt the dealers and improve the welfare of the investors. This view is different from a point made by Harris (1994) in favor of coarse price grids. He claims that a small tick size disrupts time priority, since the cost of undercutting the current quote by opportunistic "quote matchers" decreases with the tick size.

Consequences of discrete prices have been extensively investigated in the context of liquidity provision in different kinds of financial markets. The main question asked in this context is what is the impact of tick size on liquidity, which is traditionally measured by the bid-ask spread. Most of the theoretical papers do not tackle the issue of competition explicitly. Seppi (1997) and Anshuman and Kalay (1998) assume zero profits, and thus analyze the perfectly competitive case. Bernhardt and Hughson (1996), and Cordella and Foucault (1999) consider two dealer markets only, while Chordia and Subrahmanyam (1995), and Kandel and Marx (1997) assume Bertrand competition. In all these cases the full scale of competition levels from a monopoly to the perfectly competitive case cannot be considered, and thus our insights about low levels of competition are skipped. Glosten (1995) models finite levels of competition in a discrete setting. In his model, risk aversion and competition for quantities eliminate zero profits. However, he focuses on the impact of price grid refinement given a fixed level of competition, and does not investigate the interplay between the level of competition and price discreteness.[1]

Another difference between our paper and the extant financial literature on discrete prices is our approach to evaluating market quality. While there is no doubt that the cost of liquidity provision is an important market feature, we take a somewhat different route. We follow Holmstrom and Myerson (1983), and Myerson and Satterthwaite (1983), and view a market mechanism as a means for exploiting potential gains from trade. Therefore, we use ex-ante expected gains from trade (and not the spread) as a measure for the welfare of different market participants. This approach is dominant in the mechanism design literature.

Our dealer market is similar to a first price auction mechanism. This modeling technique has been introduced to the financial literature by Biais (1993). In his model the number of dealers is endogenous, and the market order is not limited to just one unit of stock. On the other hand, Biais does not study the impact of discrete prices at all (he uses continuous pricing). Moreover, in his model investors act strategically by choosing the size of the market order, while in our model the strategic behavior of investors is reflected in their decision of whether or not to submit the market order.

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[1] There is also a rich empirical literature on liquidity implications of discrete prices in financial markets. See Harris (1994, 1997) or Chakravarty, Harris and Wood (2001) for a survey of this literature. Partial emprical corroboration to our results may be found in Christie, Harris and Kandel (2001). They show that the reaction to tick size reductions in NASDAQ is associated with the level of competition, in the same directions as our model predicts.

This is important to our model because it enables us to investigate the case of a monopolistic dealer.

The strategic behavior of the investors in our model makes it similar to a mechanism known as the “buyer’s bid double auction”. This mechanism was introduced by Chatterjee and Samuelson (1983), and its welfare attributes under different levels of competition have been explored by Wilson (1985), Williams (1987), Satterthwaite and Williams (1989a,b), Williams (1991), Rustichini, Satterthwaite, and Williams (1994), Gong and McAfee (1996), and Zacharias and Williams (2001). Unfortunately, we were not able to use most of these results, because they rely on a first order approach, which is not valid in the face of discrete prices. Leininger, Linhart, and Radner (1989) show that “step function” equilibria, where agents choose to use only a finite number of prices, may exist even with continuous pricing.

The paper is organized as follows: Section 2 presents the model. In Section 3 we prove existence of non-decreasing equilibria, and define our welfare measures. Section 4 analyzes the welfare implications of discrete prices in the monopolistic case. In section 5 we discuss the implications of increasing the level of competition, and provide a convergence result. Section 5 investigates the welfare implications of discrete prices in competitive frameworks, and analyzes the rate of convergence of the welfare measures. Section 6 concludes. Technical proofs are in the Appendices.

2 Model

We model the lower half of a dealer market for a specific stock. In this lower half, dealers submit “buy” quotes anticipating a “sell” market order from investors.[2] Our market consists of $m+1$ agents ($m \geq 1$). The first m agents are designated as dealers. Each dealer is willing to buy one unit of stock. The last agent is a representative investor, this agent is endowed with one unit of stock that she is willing to sell. Each dealer has his own reservation value of the stock, which stems from different

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[2] Modeling the upper half of the market is symmetric.

inventory positions. A long inventory position implies low valuation while a short position implies high valuation.[3] Reservation values are assumed to be independent across dealers, reflecting the cross sectional independence of inventory positions.

The distribution of reservation values for each dealer is given by the cumulative distribution function F. The following assumptions are used throughout the paper:

A.1 The support of F is $[0, 1]$.

A.2 F is continuously differentiable, and has a strictly positive density function denoted by f.

A.3 Reservation values are statistically independent across dealers, and are each dealer's private information.[4]

The representative investor's reservation value is distributed according to a cumulative distribution function denoted by G. We further assume that:

A.4 The support of G is $[0, 1]$.

A.5 The investor's reservation value is her own private information, and is statistically independent from the reservation values of the m dealers.

Reservation values may take any value in $[0, 1]$, however quotes (and prices) are restricted to a finite grid. The market designer chooses an integer $n \geq 1$, such that quotes must be in the set: $P_n \equiv \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$. The minimum price variation, or tick size is $\frac{1}{n}$.

The trading mechanism is as follows: first the m dealers approach the market at once. Each one of them submits a quote in the form: "I am willing to buy one unit of stock at any price not exceeding p" where $p \in P_n$. Quotes are submitted

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[3] See Biais (1993) for a specific derivation of private valuations from inventory positions.

[4] See the "discussion and conclusions" section at the end of the paper for a discussion of the robustness of this assumption.

contemporaneously, thus dealers cannot see the quotes of other dealers. In the second stage the investor approaches the market and views all the quotes submitted by dealers. She may either submit a “market order” or quit. A market order means that the investor sells one unit of stock to the dealer who submitted the best (highest) quote. If the investor quits then no transaction is made. In case of a tie between a few dealers the winner is selected using a fair lottery. This lottery is a proxy for time priority, when the ex-ante probability for a certain dealer to be the first who submitted a quote at a certain price is uniform. Notice that since prices are discrete, ties are very likely when m is large relative to n.

We assume that agents' payoffs are linear in money transfers. Namely, suppose that a transaction occurs between an investor with reservation value $x \in [0, 1]$ and a winning dealer with reservation value $y \in [0, 1]$ at a price $p \in P_n$. The investor's profit is the price she gets minus her reservation value: $p - x$, and the winning dealer's profit is his reservation value minus the price he pays: $y - p$. All other $m - 1$ dealers earn zero. In case of no transaction all the $m + 1$ agents earn zero. All the parameters of the model, the rules of the trading mechanism as well as the distributions F and G are assumed to be common knowledge among all agents.

This setting may be viewed as a game of incomplete information by identifying the agents' reservation values as their types. A strategy for a dealer in this game is a mapping $B : [0, 1] \to P_n$, that assigns a quote $B(y)$ to any possible reservation value y. A strategy for the investor is a mapping from $\{[0, 1] \times (P_n)^m\}$ to the set {“submit a market order”, “quit”}. Thus, a strategy for the representative investor is a mapping that assigns any combination of investor's reservation values and m dealers' quotes with a decision of submitting a market order or quitting.

Potentially, the investor might have many different strategies. However, it is readily seen that submitting a market order if and only if the highest dealer's quote is at least as high as the investor's reservation value is a (weakly) dominant strategy for the investor. Namely, regardless of the dealers' strategies, the investor cannot improve her payoff over submitting a market order whenever at least one of the dealers submitted a quote that exceeds her reservation value, and quitting otherwise. To see this, denote by $b \in P_n$ the maximal dealer's quote, and denote the investor's reservation value by x. If $b < x$, submitting a market order will provide the investor with a negative payoff of $b - x$, however by quitting the investor can assure herself a payoff of zero. On the other hand, if $b \geq x$, submitting a market order will provide the investor with a non-negative payoff, which is (weakly) preferred to quitting. From now on we will assume that the representative investor always employs her dominant strategy. Our focus will lie on the quoting strategies of the m dealers given this obvious investor's strategy.

Next, we obtain the dealer's objective function. Let $B(\cdot)$ denote a dealer's strategy, and let $y \in [0,1]$, and $b \in P_n$. We denote by $\pi_{n,m}(b, y, B)$ the expected payoff to a dealer given that: (i) the dealer's reservation value is y and he submits a quote of b; (ii) the investor uses her dominant strategy; and (iii) all other $m-1$ dealers use the strategy $B(\cdot)$. $\pi_{n,m}(b, y, B)$ is given by $y - b$ (the payoff in case of winning the transaction) times the probability that a transaction really occurs and that our specific dealer is the one who wins it. Formally:

$$\pi_{n,m}(b, y, B) = (y - b)G(b) \Pr \left\{ \begin{array}{l} \text{The dealer wins against the other } m-1 \\ \text{dealers, given that they all use the} \\ \text{strategy } B(\cdot) \end{array} \right\} \tag{1}$$

In a symmetric Nash equilibrium each dealer's strategy maximizes his expected payoff, given that the other dealers use the same strategy.[5] Therefore we define: A symmetric, pure strategy Nash equilibrium in this game is given by a dealer's strategy $B : [0,1] \to P_n$, such that for almost all $y \in [0,1]$:[6]

$$B(y) \in \arg\max_{b \in P_n} \pi_{n,m}(b, y, B) \tag{2}$$

Ex-post efficiency requires that if the investor values the stock less than one of the

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[5] For simplicity we confine our attention in this paper to symmetric equilibria only.

[6] In this definition as well as in the remainder of the paper, the term "for almost all" pertains to the probability measure induced by the buyer's distribution F.

dealers, then she should sell it to the dealer with the highest valuation. Unfortunately, due to Myerson and Satterthwaite (1982) no bilateral trade mechanism that allows for individual rationality and incentive compatibility is ex-post efficient. Our mechanism, is no different. Indeed, a dealer will never submit a buy-quote higher than his reservation value, however, typically dealers will submit buy-quotes lower than their reservation values because of strategic behavior. This behavior might prevent trade in situations where gains from trade exist.

While this inefficiency effect is common also in the continuous pricing literature, price discreteness worsens the situation. The reason is that there are only finitely many prices whereas there is a continuum of types. Typically, the highest quote will be submitted by multiple dealers with different types. Since the winner is selected by chance, there is no special reason why the dealer with the highest reservation value will be selected.

We conclude that inefficiency in our market stems from two sources: (i) strategic behavior of dealers, we call this the "strategic effect"; and (ii) clustering of dealers with different types on the same quote price. We call this the "clustering effect". In order to achieve maximum efficiency both these effects must be minimized.

Reducing the strategic effect may be obtained by causing the dealers to submit quotes as close as possible to their reservation values. In order to formalize this we define the "honest strategy", $H_n : [0,1] \to P_n$ by:

$$H_n(y) \equiv \begin{cases} \frac{k}{n} & \frac{k}{n} \leq y < \frac{k+1}{n} \quad for\ k = 0, ..., n-1 \\ 1 & y = 1 \end{cases}$$

A dealer that uses the honest strategy H_n, shades his value as little as possible, given that he may only submit quotes in P_n. There is no way that the market designer can force dealers to use the honest strategy, since whenever $n > 2$ it is not an equilibrium.[7]

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[7] Suppose $n > 2$ and $m \geq 1$, and let $k \in \{2, ..., n-1\}$. Suppose that dealer 1 has reservation value $y \in (\frac{k}{n}, \frac{k+1}{n})$, and the other $m-1$ dealers use H_n. As y gets closer to $\frac{k}{n}$, the expected profit to dealer 1 from submitting $\frac{k}{n}$ tends to zero. However, if he deviates and submits $\frac{k-1}{n}$, he gets a positive expected profit. This shows that H_n cannot be an equilibrium.

Diminishing the clustering effect may be obtained by diminishing the tick size (increasing n), and hence lowering the probability of bidding ties by different types. The tick size is definitely a decision variable of the market designer, and he can choose it as he likes. However, a priori it is not clear what is the cross impact of decreasing the tick size on the strategic effect, because decreasing the tick size changes the set of allowable prices for dealers, and hence affects their bidding strategies. It is commonly believed that decreasing the tick size improves the position of the investors, and worsens the position of dealers. This belief has led to the recent decimalization in US stock markets. We will show that this common wisdom is sensitive to the level of competition among dealers.

3 Non-decreasing equilibria

Following an approach introduced by Athey (2001) we focus in this paper on non-decreasing equilibria, i.e. equilibrium bidding strategies $B(y)$ that are non-decreasing in the dealer's reservation value y.

Since P_n contains only a finite number of actions, any non-decreasing strategy is just a step function with a finite number of steps. Therefore, we may identify any non-decreasing strategy with its jump points, i.e. the points on the unit interval at which the dealer jumps from one quote in P_n to another one. This is done using the following definition: We say that a vector $\sigma^B \equiv (\sigma_0^B, \sigma_1^B, ..., \sigma_{n+1}^B) \in \mathbb{R}^{n+2}$ represents a dealer's strategy B, if for all $h \in \{0, ..., n+1\}$, $\sigma_h^B = \inf\{y \in [0,1] : B(y) \geq \frac{h}{n}\}$ whenever there is some $h' \geq h$ such that $B(y) = \frac{h'}{n}$ on an open interval of $[0,1]$, and $\sigma_h^B = 1$ otherwise. The space of all non-decreasing strategies' representations for a dealer is:

$$\Sigma \equiv \{\sigma \in [0,1]^{n+2} : 0 = \sigma_0 \leq \sigma_1 \leq ... \leq \sigma_n \leq \sigma_{n+1} = 1\}$$

This is a compact subset of $[0,1]^{n+2}$.

To exemplify this definition consider the case $n = 6$, and let $B^*(\cdot)$ be the non-decreasing strategy defined by:

$$B^*(y) = \begin{cases} 0 & 0 \le y < \frac{1}{4} \\ \frac{2}{6} & \frac{1}{4} \le y < \frac{3}{4} \\ \frac{4}{6} & \frac{3}{4} \le y < 1 \\ 1 & y = 1 \end{cases} \tag{3}$$

Then $\sigma^{B^*} = (0, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, 1, 1, 1) \in \mathbb{R}^8$.

Notice that for any pair of non-decreasing strategies B_1 and B_2 we have $B_1 = B_2$ almost everywhere on $[0, 1]$ if and only if $\sigma^{B_1} = \sigma^{B_2}$. Also, a sequence of non-decreasing strategies $\{B_m\}_{m=1}^{\infty}$ converges to a non-decreasing strategy B almost everywhere on $[0, 1]$ if and only if $\lim\limits_{m\to\infty} \sigma^{B_m} = \sigma^B$. Thus, the question of convergence of non-decreasing strategies may be reduced to a question of convergence of vectors in the finite dimensional Euclidean space $\mathbb{R}^{n+2}$.

The following notation is useful: Given a non-decreasing strategy $B : [0, 1] \to P_n$, that is represented by a vector $\sigma^B \in \mathbb{R}^{n+2}$, we denote: $\Delta_n(B) \equiv \{\frac{k}{n} \in P_n : \sigma^B_{k+1} > \sigma^B_k\}$. Thus, $\Delta_n(B)$ is the subset of P_n, of quote prices that are submitted with positive probability according to $B(\cdot)$. To illustrate this notation observe that $\Delta_n(B^*) = \{0, \frac{2}{6}, \frac{4}{6}\}$ where B^* is the strategy defined by (3).

In order to further investigate the nature of equilibria in multi-dealer environments we shall re-phrase the objective function of a specific dealer given by Equation (1) in a more explicit form. Let $B(\cdot)$ be any non-decreasing strategy for dealers, and let $y \in [0, 1]$ and $b = \frac{k}{n} \in P_n$. Recall that $\pi_{n,m}(b, y, B)$ denotes the expected profit to a specific dealer given that his reservation value is y, he bids b, and all the other $m - 1$ dealers use the strategy $B(\cdot)$.

If $b \in \Delta_n(B)$ then (1) may be written as follows:

$$\pi_{n,m}(b, y, B) = (y - \frac{k}{n})G(\frac{k}{n}) \times \tag{4}$$

$$\times (F(\sigma^B_{k+1}))^{m-1} \sum_{r=0}^{m-1} \binom{m-1}{r} \left(\frac{F(\sigma^B_k)}{F(\sigma^B_{k+1})}\right)^r \left(\frac{F(\sigma^B_{k+1}) - F(\sigma^B_k)}{F(\sigma^B_{k+1})}\right)^{m-r-1} \cdot \frac{1}{m-r}$$

The term: $(y - \frac{k}{n})G(\frac{k}{n})$ is the profit of the dealer in case of winning, times the probability that the investor chooses to submit a market order. To see the logic behind the other terms, notice that in order for a dealer who bids $b = \frac{k}{n}$ to win the stock over the other $m-1$ dealers, there must not be any dealer with reservation value equal to or higher than σ_{k+1}^B. The probability for this event is $(F(\sigma_{k+1}^B))^{m-1}$. Conditional on this event, the $m-1$ other dealers may be divided such that r of them ($r \in \{0, 1, ..., m-1\}$) have reservation values in $[0, \sigma_k^B)$ and $m-r-1$ of them have reservation values in $[\sigma_k^B, \sigma_{k+1}^B)$. The r former dealers will surely loose to our dealer since their bid is less than $\frac{k}{n}$, while the $m-r-1$ latter dealers will tie with our specific dealer, hence he has a probability of $\frac{1}{m-r}$ to win against them. Going over all r from 0 to $m-1$, and taking into account all the possible divisions of the $m-1$ dealers to two groups of r and $m-r-1$ yields (4).

After some simplifications and using the binomial formula Equation (4) reduces to:

$$\pi_{n,m}(\frac{k}{n}, y, B) = \frac{1}{m}(y - \frac{k}{n})G(\frac{k}{n})\frac{(F(\sigma_{k+1}^B))^m - (F(\sigma_k^B))^m}{F(\sigma_{k+1}^B) - F(\sigma_k^B)} \tag{5}$$

If $b = \frac{k}{n} \notin \Delta_n(B)$, then $\sigma_k^B = \sigma_{k+1}^B$ and $\frac{k}{n}$ is submitted with zero probability according to $B(\cdot)$. In this case there is probability zero that submitting a quote of $\frac{k}{n}$ will result in a tie. Hence in order to win the stock, the reservation values of the other $m-1$ dealers must be lower than σ_k^B. Therefore, in this case:

$$\pi_{n,m}(b, y, B) = (y - \frac{k}{n})G(\frac{k}{n})F(\sigma_k^B)^{m-1} \tag{6}$$

Having established the expressions for the dealers' objective function we can now show that a non-decreasing, symmetric equilibrium always exists in our model.[8] Our existence proof relies on a result by Athey (2001). This result asserts that if the objective function satisfies the single crossing property of Milgrom and Shannon (1994), and the set of potential actions is finite, a non-decreasing pure strategy Nash equilibrium exists. We must, however, introduce some mild changes to Athey's proofs

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[8] This fact should be appreciated since in general, existence of equilibria in games of incomplete information with a continuum of types is an open question. The problem is even harder in games in which the objective function is discontinuous in the agents' actions, as is the case in this model (see also Reny 1999). Williams (1991) has proved existence of equilibrium in a similar model with continuous prices. His proof uses a differential equations technique which is not applicable in our discrete setting.

since she does not show existence of *symmetric* equilibria, while our results apply to *symmetric* equilibria only. The following proposition states the existence result. Details of the proof as well as a formal statement of Milgrom and Shannon's single crossing property, and Athey's existence result may be found in Appendix A.

Proposition 1 *Consider a market with a tick size of $\frac{1}{n}$ and m dealers ($n, m \geq 1$), and suppose the investor uses her dominant strategy. There exists a non-decreasing, symmetric, pure strategy Nash equilibrium.*

In the following sections we investigate the welfare properties of this kind of non-decreasing equilibria. We gauge the ex-ante gains from trade in our mechanism, i.e. the expected gains from trade calculated before any market participant knows his reservation value. This is done as follows.

Consider a market with a tick size of $\frac{1}{n}$ and m dealers, and let $B_{n,m}(\cdot)$ be a non-decreasing, pure strategy, symmetric Nash equilibrium, represented by $\sigma^{B_{n,m}} \in \mathbb{R}^{n+2}$. Let $F_{(m)}(y)$ denote the distribution of the highest among m draws of dealers' reservation values.[9] We have: $F_{(m)}(y) = (F(y))^m$. The investor's ex-ante expected profit given this equilibrium is:

$$\Gamma_{n,m}^{investor}(B_{n,m}) \equiv \int_{y=0}^{1} \int_{x=0}^{B_{n,m}(y)} (B_{n,m}(y) - x) dG(x) dF_{(m)}(y)$$

If we denote: $\gamma(t) \equiv \int_0^t G(x)dx$, then by applying integration by parts, the ex-ante profits of the investor may be written as follows:

$$\begin{aligned} \Gamma_{n,m}^{investor}(B_{n,m}) &= \int_{y=0}^{1} \gamma(B_{n,m}(y)) dF_{(m)}(y) \\ &= \sum_{k=1}^{n-1} \gamma(\frac{k}{n}) \left[(F(\sigma_{k+1}^{B_{n,m}}))^m - (F(\sigma_{k}^{B_{n,m}}))^m \right] \end{aligned} \tag{7}$$

The ex-ante profits to the group of m dealers given the equilibrium $B_{n,m}$ is:

$$\Gamma_{n,m}^{dealers}(B_{n,m}) \equiv \int_{y=0}^{1} \pi_{n,m}(B_{n,m}(y), y, B_{n,m}) dF(y)$$

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[9] Also known as the mth order statistic.

Using Equation (5) this can be written as follows:[10]

$$\Gamma^{dealers}_{n,m}(B_{n,m}) = \sum_{k=1}^{n-1} G(\frac{k}{n}) \frac{(F(\sigma^{B_{n,m}}_{k+1}))^m - (F(\sigma^{B_{n,m}}_{k}))^m}{F(\sigma^{B_{n,m}}_{k+1}) - F(\sigma^{B_{n,m}}_{k})} \int_{\sigma^{B_{n,m}}_{k}}^{\sigma^{B_{n,m}}_{k+1}} (y - \frac{k}{n}) f(y) dy \qquad (8)$$

The total gains from trade in the market are defined by: $\Gamma^{total}_{n,m}(B_{n,m}) \equiv \Gamma^{investor}_{n,m}(B_{n,m}) + \Gamma^{dealers}_{n,m}(B_{n,m})$.

In general, we cannot calculate equilibria for arbitrary values of m and n, and distribution functions G and F. We can always do so, however, when $m = 1$ (a monopolistic dealer). We can also show that equilibria converge to the honest strategy as m increases to infinity. Thus, in the next sections we shall investigate these two extreme case. Intermediate levels of competition feature a gradual movement from the former case to the latter, and are analyzed by evaluating the rate of convergence to the perfectly competitive outcome.

4 Discrete prices with a monopolistic dealer

Consider the case of a monopolistic dealer ($m = 1$). We would like to gauge the effect of a change in tick size on the welfare measures in this market. Suppose that the tick size is $\frac{1}{n}$. Since the investor uses her dominant strategy, the dealer's equilibrium strategy $B_{n,1}(y)$ solves:

$$B_{n,1}(y) \in \arg\max_{b \in P_n} \pi_{n,1}(b, y, B_{n,1}) \equiv (y - b)G(b)$$

Existence of equilibrium in this case is trivial since the set of actions is finite and the dealer just chooses the action that maximizes his expected profit for each possible type y, given that the investor plays her dominant strategy. Also, it is easy to see that in this case, equilibrium is unique up to changes of zero measure.[11]

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[10] Notice that for the sake of calculating $\Gamma^{buyers}_{n,m}(B_{n,m})$ we can dispense with equation (6) because the probability of the event: $\{y : B_{n,m}(y) \notin \Delta_n(B_{n,m})\}$ is zero by definition.

[11] Let $b_1, b_2 \in P_n$, $y_0 \in [0,1]$ and suppose that $(y_0 - b_1)G(b_1) = (y_0 - b_2)G(b_2)$. By A.4 the support of G is $[0,1]$, therefore: $G(b_1) \neq G(b_2)$ and hence y_0 is the only point in $[0,1]$ for which that equality holds. Since P_n is finite we obtain that equality holds for at most finitely many points in $[0,1]$, hence equilibrium is unique up to changes of zero measure.

Let $\eta \geq 1$ be an integer and let $\tilde{n} = n\eta$. Observe that $P_{\tilde{n}}$ is a refinement of P_n in the sense that any price that is feasible when the tick size is $\frac{1}{n}$ is also feasible when the tick size is $\frac{1}{\tilde{n}}$. In this section we restrict our attention to this kind of "refinement" changes in tick size, which is pretty prevalent in reality (e.g. a change from a tick size of $\$\frac{1}{8}$ to $\$\frac{1}{16}$). Other kinds of changes would introduce some pathologies to the analysis. In the next proposition we characterize the unique equilibrium in this setting.

Proposition 2 *Consider a market with a monopolist dealer ($m = 1$), and a tick size of $\frac{1}{n}$. The unique (up to changes of zero measure) equilibrium $B_{n,1}(\cdot)$ is represented by the vector $\sigma^{B_{n,1}} \in \mathbb{R}^{n+2}$ where: $\sigma_0^{B_{n,1}} = 0$, $\sigma_n^{B_{n,1}} = \sigma_{n+1}^{B_{n,1}} = 1$ and for all $k = 1, ..., n-1$:*

$$\sigma_k^{B_{n,1}} = \min(\frac{k}{n} + \frac{1}{n} \cdot \frac{G(\frac{k-1}{n})}{G(\frac{k}{n}) - G(\frac{k-1}{n})}, 1) \tag{9}$$

The proof of this proposition is immediate. It follows from the fact that for $y = \sigma_k^{B_{n,1}}$, the dealer is indifferent between submitting a quote of $\frac{k}{n}$ and a quote of $\frac{k-1}{n}$. Therefore: $(\sigma_k^{B_{n,1}} - \frac{k}{n})G(\frac{k}{n}) = (\sigma_k^{B_{n,1}} - \frac{k-1}{n})G(\frac{k-1}{n})$. Solving for $\sigma_k^{B_{n,1}}$ yields (9).

Notice that the equilibrium does not depend on F at all. This is so because the dealer maximizes $\pi_{n,1}(\cdot)$ which does not depend on F due to lack of competition.

Our first observation regarding welfare is that decreasing the tick size will *always* improve the position of the monopolistic dealer. This is so because by refining the grid of potential prices we enrich the set of actions he is allowed to take, and since there is no competitive pressure he uses this richness, and his "first mover advantage" to increase his payoff. This observation is formalized in the next proposition.

Proposition 3 *Consider a market with a monopolistic dealer, and let $\eta > 1$ be an integer. Suppose $B_{n,1}(\cdot)$ and $B_{\tilde{n},1}(\cdot)$ are the equilibrium strategies when the tick size is $\frac{1}{n}$ and $\frac{1}{\tilde{n}} \equiv \frac{1}{\eta n}$ respectively. The expected profits to the dealer given a tick size of $\frac{1}{\tilde{n}}$ are larger than when the tick size is $\frac{1}{n}$. Namely: $\Gamma_{n,1}^{dealer}(B_{n,1}) \leq \Gamma_{\tilde{n},1}^{dealer}(B_{\tilde{n},1})$.*

Proof. From Equation (8), the dealer's expected profits are given by:

$$\Gamma_{n,1}^{dealer}(B_{n,1}) \;=\; \int_{y=0}^{1} \pi_{n,1}(B_{n,1}(y), y, B_{n,1}) f(y) dy = \int_{y=0}^{1} (y - B_{n,1}(y)) G(B_{n,1}(y)) dy$$

$$\Gamma^{dealer}_{\tilde{n},1}(B_{\tilde{n},1}) \;=\; \int_{y=0}^{1} \pi_{n,1}(B_{\tilde{n},1}(y), y, B_{\tilde{n},1}) f(y)dy = \int_{y=0}^{1} (y - B_{\tilde{n},1}(y))G(B_{\tilde{n},1}(y))dy$$

Since $P_{\tilde{n}}$ is finer than P_n, $(y - B_{n,1}(y))G(B_{n,1}(y)) \leq (y - B_{\tilde{n},1}(y))G(B_{\tilde{n},1}(y))$ for all y. Therefore: $\Gamma^{dealer}_{\tilde{n},1}(B_{\tilde{n},1}) \geq \Gamma^{dealer}_{n,1}(B_{n,1})$. ■

Although this result is almost trivial, it forms the basis for the *main theme* of this paper. It shows that in the absence of competition, dealers prefer fine grids, since they enable them to make better use of their "first mover advantage". It is, however, our duty to show that this result carries on to markets with higher levels of competition. We will deal with this issue in Section 6.

Contrary to this decisive result, the effect of a decrease in tick size on the investor, and on total welfare is ambiguous. To capture this point, notice that the investor's welfare may increase following two kinds of events: (i) a transfer of welfare from the dealer to the investor; and (ii) an increase in the probability of transaction. If the probability of transaction were constant, any decrease in the welfare of the dealer would be immediately translated to an increase in the welfare of the investor (a zero sum game). Thus, following Proposition 3, any decrease in tick size would mean a lower welfare for the representative investor. However, the probability of transaction varies following a change in tick size; it might increase or decrease. Thus, a decrease in tick size in the monopolistic dealer case has an ambiguous effect on the investor and on total welfare. Notice, however, that a decrease in tick size in this case always has a negative effect on the investor because it benefits the dealer. Only in cases were the increase in probability of trade compensates this negative effect will the investor be better of with a smaller tick size. This implies that in the monopolistic dealer case, investors have a tendency towards larger ticks. The extent of this tendency depends on the distribution of reservation values.

We shall now provide two examples to demonstrate that in this monopolistic framework (i) decreasing the tick size might have no effect at all on total welfare and (ii) decreasing the tick size might even impair total welfare. One can also easily devise examples where decreasing the tick size is welfare improving. Thus, the impact of refining the grid of prices on market welfare is counterintuitively ambiguous in this setting.

Example 1 *A large tick size might be highly efficient.*

Assume that F and G are uniformly distributed over $[0,1]$, and let n be an even integer. By Proposition 2 it follows that the unique equilibrium $B_{n,1}(\cdot)$ is given by:

$$B_{n,1}(y) = \begin{cases} \frac{k}{n} & \frac{2k-1}{n} \leq y < \frac{2k+1}{n} \qquad for\ k = 0, ..., \frac{n}{2} - 1 \\ \frac{1}{2} & \frac{n-1}{n} \leq y \leq 1 \end{cases}$$

Given the equilibrium $B_{n,1}(\cdot)$, we can calculate the welfare measures as follows:[12]

$$\begin{aligned} \Gamma_{n,1}^{dealer}(B_{n,1}) &= \sum_{k=1}^{\frac{n}{2}-1} \int_{y=\frac{2k-1}{n}}^{\frac{2k+1}{n}} \int_{x=0}^{\frac{k}{n}} (y - \frac{k}{n}) dxdy + \int_{y=\frac{n-1}{n}}^{1} \int_{x=0}^{\frac{1}{2}} (y - \frac{1}{2}) dxdy = \\ &= \frac{1}{12} - \frac{1}{12n^2} \\ \Gamma_{n,1}^{investor}(B_{n,1}) &= \sum_{k=1}^{\frac{n}{2}-1} \int_{y=\frac{2k-1}{n}}^{\frac{2k+1}{n}} \int_{x=0}^{\frac{k}{n}} (\frac{k}{n} - x) dxdy + \int_{y=\frac{n-1}{n}}^{1} \int_{x=0}^{\frac{1}{2}} (\frac{1}{2} - x) dxdy \\ &= \frac{1}{24} + \frac{1}{12n^2} \\ \Gamma_{n,1}^{total}(B_{n,1}) &= \Gamma_{n,1}^{dealer}(B_{n,1}) + \Gamma_{n,1}^{investor}(B_{n,1}) = \frac{1}{8} \end{aligned}$$

These calculations suggest that a decrease in tick size increases the gains of the dealer as predicted by Proposition 3, decreases the gains to the investor, and does not affect total gains at all. Indeed, total gains from trade are equal to $\frac{1}{8}$ regardless of the tick size. This result is striking since a very coarse grid (say $n = 2$) yields the same ex-ante gains from trade as any finer grid, which are identical to the ex-ante welfare that results in the continuous pricing case.

Example 2 *Decreasing the tick size might impair total welfare.*

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[12] In the following calculations we use the identity: $1^2 + 2^2 + ... + r^2 = \frac{r(r+1)(2r+1)}{6}$, for any positive integer r.

Let us keep the assumption that G is uniform and suppose that f is given by:[13]

$$f(y) = \begin{cases} 0 & 0 \leq y < \frac{1}{2} \\ 4 & \frac{1}{2} \leq y < \frac{3}{4} \\ 0 & \frac{3}{4} \leq y \leq 1 \end{cases}$$

Consider the equilibrium strategies in the case $n = 2$ and $n = 4$, denoted by $B_{2,1}$ and $B_{4,1}$ respectively. Since equilibrium behavior does not depend on F, these strategies are the same as in Example 1. However, F does influence our welfare measures. Calculation shows for the case $n = 2$ that: $\Gamma_{2,1}^{dealer}(B_{2,1}) = \frac{1}{16}$, $\Gamma_{2,1}^{investor}(B_{2,1}) = \frac{1}{8}$ and $\Gamma_{2,1}^{total}(B_{2,1}) = \frac{3}{16}$. Whereas for the case $n = 4$ we obtain: $\Gamma_{4,1}^{dealer}(B_{4,1}) = \frac{3}{32}$, $\Gamma_{4,1}^{investor}(B_{4,1}) = \frac{1}{32}$ and $\Gamma_{4,1}^{total}(B_{2,1}) = \frac{1}{8}$. Thus, by moving from a tick size of $\frac{1}{2}$ to a tick size of $\frac{1}{4}$, total ex-ante welfare decreases from $\frac{3}{16}$ to $\frac{1}{8}$ (a decrease of 33% in total welfare). In this example, grid refinement decreases the probability of execution. This causes the decline in total welfare.

To summarize: in the case of a monopolistic dealer, a decrease in tick size improves the position of the dealer, and has an ambiguous impact on the representative investor with some tendency towards larger spreads. In what follows we claim that these welfare patterns hold when m is larger than 1 but is sufficiently small.

5 Increasing the level of competition

Our model features two "levers" that can be pulled: the tick size $\frac{1}{n}$, and the level of competition m. The tick size is purely a decision of the market designer, while the level of competition should be determined endogenously. Since in our model m is given exogenously, it would be fruitful the study the effect of competition on equilibrium bidding strategies. We will show that as the level of competition increases, dealers are forced to submit higher quotes. Actually, when m tends to infinity, any sequence

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[13] This f violates the assumption that the density function is strictly positive and continuous. However, this can by easily amended by increasing the mass over the zeroed intervals slightly, and decreasing the mass over the interval $[\frac{1}{2}, \frac{3}{4})$ accordingly, in a way that will assure continuity as well. Continuity of the efficiency measures assures us that our results will keep holding when we use the amended density.

of equilibrium strategies tends to the honest strategy H_n.

Proposition 4 *Assume a tick size of* $\frac{1}{n}$. *Let* $\{B_{n,m}\}_{m=1}^{\infty}$ *be any sequence of non-decreasing equilibrium strategies in markets with* m *dealers, then* $\lim\limits_{m\to\infty} B_{n,m} = H_n$ *almost everywhere on* $[0,1]$.

A formal proof of Proposition 4 is given in Appendix A. We provide here an outline of the proof. We proceed in two steps. The first step is to show that if a sequence of non-decreasing equilibrium strategies $\{B_{n,m}\}$ in markets with m dealers converges to B_n, then $B_n = H_n$ almost everywhere. The driving force behind this assertion is as follows. Suppose on the contrary that $B_n \neq H_n$ on a set of positive measure. It follows that dealers find it optimal to act strategically and shade their values downwards. Namely, there exists $k \in \{1, 2, ..., n-1\}$, $h \in \{0, ..., k-1\}$ and a non-null set $Q \subset (\frac{k}{n}, \frac{k+1}{n})$ such that for all $y \in Q$ and m large enough $B_m(y) = \frac{h}{n}$. Thus for all m large enough and $y \in Q$: $\pi_{n,m}(\frac{h}{n}, y, B_{n,m}) \geq \pi_{n,m}(\frac{k}{n}, y, B_{n,m})$. If we assume $\frac{k}{n} \in \Delta_n(B_m)$ for all large enough m (this assumption is justified in Appendix A) then using Equation (4) we obtain for all $y \in Q$ and m large enough:

$$(y-\frac{h}{n})G(\frac{h}{n})\frac{(F(\sigma_{h+1}^{B_{n,m}}))^m-(F(\sigma_h^{B_{n,m}}))^m}{F(\sigma_{h+1}^{B_{n,m}})-F(\sigma_h^{B_{n,m}})} \geq (y-\frac{k}{n})G(\frac{k}{n})\frac{(F(\sigma_{k+1}^{B_{n,m}}))^m-(F(\sigma_k^{B_{n,m}}))^m}{F(\sigma_{k+1}^{B_{n,m}})-F(\sigma_k^{B_{n,m}})}$$

Or equivalently:

$$\frac{(y-\frac{h}{n})G(\frac{h}{n})}{(y-\frac{k}{n})G(\frac{k}{n})} \geq \frac{\frac{(F(\sigma_{k+1}^{B_{n,m}}))^m-(F(\sigma_k^{B_{n,m}}))^m}{F(\sigma_{k+1}^{B_{n,m}})-F(\sigma_k^{B_{n,m}})}}{\frac{(F(\sigma_{h+1}^{B_{n,m}}))^m-(F(\sigma_h^{B_{n,m}}))^m}{F(\sigma_{h+1}^{B_{n,m}})-F(\sigma_h^{B_{n,m}})}}$$

The l.h.s of this inequality does not depend on m, while we can show that the r.h.s tends to infinity as m increases, regardless of the specific form of the strategies $B_{n,m}$.[14] Thus, this inequality cannot hold for large enough m - a contradiction. Intuitively, as m increases, dealers find it optimal to increase their bids because it increases their odds of winning the stock, to a larger extent than it lowers their payoff in case of

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[14] All that is needed is that the sequence of strtegies $\{B_{n,m}\}$ converges.

winning. Thus, for large enough m submitting as high a bid as possible becomes optimal.

The second step of the proof is to show that any sequence of equilibrium strategies $\{B_{n,m}\}$ in markets with m dealers does indeed converge. In order to establish this result we refer to the representation $\sigma^{B_{n,m}} \in \mathbb{R}^{n+2}$ of the non-decreasing strategy $B_{n,m}$. The sequence of representations $\{\sigma^{B_{n,m}}\}_{m=1}^{\infty}$ lies in Σ (the set of all possible representations) which is a compact subset of a finite dimensional Euclidean space. By the previous step, any convergent subsequence of $\{\sigma^{B_{n,m}}\}$ converges to σ^{H_n}. This in turn, together with the compactness of Σ implies that $\{\sigma^{B_{n,m}}\}$ converges to σ^{H_n}. Therefore, $B_{n,m} \to H_n$ almost everywhere, as required.

The convergence result is not very handy by itself. As we noted before, different markets are characterized by different levels of competition, hence we would like to get an estimate of how far is the equilibrium form the honest strategy H_n, given any finite level of competition m. The following proposition provides a bound on the probability that dealers will submit dishonest quotes. Since $B_{n,m} \to H_n$ almost everywhere, this probability tends to zero.

Proposition 5 *Assume a market with a tick size of* $\frac{1}{n}$*, and consider a sequence of non-decreasing equilibrium strategies* $\{B_{n,m}\}$ *in markets with* m *dealers. The probability of submitting a dishonest quote by a dealer is* $O(\frac{1}{m})$*. Furthermore:* $\sigma_h^{B_{n,m}} = \frac{h}{n} + \frac{1}{n} \cdot O(\frac{1}{m})$, $h \in \{0, 1, ..., n-1\}$.[15]

Figure **??** demonstrates this proposition. It presents an equilibrium strategy in the case $n = 4$. The shaded areas on the horizontal axis designate types for which the strategy specifies a dishonest quote. According to Proposition 5, the sum of the measures of these areas is $O(\frac{1}{m})$.

Insert Figure 1 about here

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[15] We say that a function $f(m)$ is $O(\frac{1}{m})$ if there exists a constant κ, such that for large enough m, $|f(m)| \leq \frac{1}{m}$.

6 Discrete prices in competitive markets

As the number of dealers increases, they are forced to submit honest quotes for all but a small set of types. In section 4 we saw that decreasing the tick size in the monopolistic case may cause some odd results. In this section we investigate the other limiting case, of decreasing the tick size in a perfectly competitive market. Then we examine the extent to which these results may be applied to markets with limited levels of competition. We demonstrate that if competition is lax, the results of the monopolistic case may apply.

Let us fix a tick size of $\frac{1}{n}$, and consider any sequence of equilibrium strategies $\{B_{n,m}\}$ in markets with m dealers. We denote: $\Gamma^{investor}_{n,\infty} \equiv \lim_{m\to\infty} \Gamma^{investor}_{n,m}(B_{n,m})$, $\Gamma^{dealers}_{n,\infty} \equiv \lim_{m\to\infty} \Gamma^{dealers}_{n,m}(B_{n,m})$, and $\Gamma^{total}_{n,\infty} = \Gamma^{investor}_{n,\infty} + \Gamma^{dealers}_{n,\infty}$.[16] These measures gauge the expected gains from trade to each group and to the market as a whole, when the market becomes perfectly competitive. In the following proposition, these measures are used in order to study the division of surplus in perfectly competitive markets. Before stating it, the following notation is needed. Denote: $\mu_G \equiv \int_0^1 x dG(x)$. This is the ex-ante expected value of the stock to the representative investor.

Proposition 6 *Consider a market with a tick size of $\frac{1}{n}$. Let $\{B_{n,m}\}$ be any sequence of non-decreasing equilibrium strategies in markets with m dealers. The following holds:*

1. $\Gamma^{investor}_{n,\infty} = 1 - \mu_G - \frac{1}{n} + O(\frac{1}{n^2})$
2. $\Gamma^{dealers}_{n,\infty} = \frac{1}{2n} + O(\frac{1}{n^2})$
3. $\Gamma^{total}_{n,\infty} = 1 - \mu_G - \frac{1}{2n} + O(\frac{1}{n^2})$

The intuition behind this result is as follows. As the number of dealers increases, they are forced to submit honest quotes. Thus, the winning dealer gets the stock

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[16] In what follows we show that these limits do exist, and thus justify this notation.

and pays his reservation value minus an amount that is between zero and $\frac{1}{n}$. On average, the winning dealer gets a "discount" of approximately $\frac{1}{2n}$. This discount is the dealers' last resort, since it is the only source of profits for them in a competitive market. The profits of the dealers are, thus, approximately proportionate to the tick size. It follows that a positive tick size serves as a *coordination device* for dealers, and enables them to shade their bids even in the face of severe competition.

As for the investor, since the number of dealers tends to infinity, the highest quote is likely to be $1 - \frac{1}{n}$. Therefore, the expected gains to the investor are approximately this quote minus the expected reservation value of the investor: $1 - \frac{1}{n} - \mu_G$. Finally, since the dealers gain one half of tick size while the investor looses a whole tick size, decreasing the tick size is welfare improving. This improvement reflects the higher probability of execution following a decrease in tick size.

While this result seems plausible, there is a problem in applying it to less then perfectly competitive markets. The reason is that the welfare measures converge gradually to the perfectly competitive outcome. For low levels of competition, the welfare measures tend to resemble the monopolistic case rather than the perfectly competitive case. Thus, decreasing the tick size helps the dealers and has an ambiguous effect on the representative investor, and on total welfare. Only for sufficiently large levels of competition a decrease in tick size has the same implications as in the perfectly competitive case. The next example illustrates this point.

Example 3 *Two dealers with uniformly distributed reservation values*

Suppose $m = 2$, and F and G are uniform. If $n = 2$ the unique equilibrium is given by:

$$B_{2,2}(y) = \begin{cases} 0 & 0 \leq y < \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \leq y \leq 1 \end{cases}$$

The welfare measures in this case are: $\Gamma_{2,2}^{investor} = 0.0938$ and $\Gamma_{2,2}^{dealers} = 0.0781$.

If $n = 4$, calculation shows that the unique equilibrium is:

$$B_{4,2}(y) = \begin{cases} 0 & 0 \leq y < \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \leq y < 0.5897 \\ \frac{2}{4} & 0.5897 \leq y \leq 1 \end{cases}$$

The welfare measures in this case are: $\Gamma_{4,2}^{investor} = 0.0904$ and $\Gamma_{4,2}^{dealers} = 0.1083$.

We see that a decrease in tick size decreases the profit of the investor by 3.6%, and increases the profits of the dealers by 38.7%. Thus, despite the competition between the two dealers, decreasing the tick size is beneficial for them, and hurts the investor.

Investigation of the rate of convergence to the competitive outcome reveals that this phenomenon is even intensified. This is so because the rate of convergence to the competitive outcome decreases as the grid of prices becomes finer. Thus, when we decrease the tick size, a higher level of competition is needed to assure behavior that is similar to the perfectly competitive case. This result is stated formally in the following proposition.

Proposition 7 *Consider a market with a tick size of $\frac{1}{n}$, and let $\{B_{n,m}\}$ be any sequence of non-decreasing equilibrium strategies in markets with m dealers. The following holds:*

1. $\Gamma_{n,\infty}^{investor} - \Gamma_{n,m}^{investor}(B_{n,m}) = O\left((F(1-\frac{1}{n}))^m\right)$
2. $\Gamma_{n,m}^{dealers}(B_{n,m}) - \Gamma_{n,\infty}^{dealers} = O\left((F(1-\frac{1}{n}))^m\right)$
3. $\Gamma_{n,m}^{total}(B_{n,m}) - \Gamma_{n,\infty}^{total} = O\left((F(1-\frac{1}{n}))^m\right)$

The intuition behind this result is as follows. As the number of dealers increases, it becomes highly likely that at least one of the dealers has a reservation value that is higher than $1 - \frac{1}{n}$. By Proposition 4, $\sigma_{n-1}^{B_{n,m}} \to 1 - \frac{1}{n}$, therefore it becomes highly likely that the winning quote will be $1 - \frac{1}{n}$. Given m dealers, the probability that this is not the case is approximately $(F(1-\frac{1}{n}))^m$. Thus, the difference between the welfare measures in the perfectly competitive case to those measures given a finite number of dealers is approximately the probability that a quote smaller than $1 - \frac{1}{n}$ will win the stock. This probability increases the smaller is the tick size.

To see the difficulty arising from this result, let $n_1 > n_2 > 0$ be integers. We would like to compare the welfare in two markets with tick sizes of $\frac{1}{n_1}$ and $\frac{1}{n_2}$, and m dealers. Consider for example the profits of the investor. We have:

$$\Gamma^{investor}_{n_1,m} = \Gamma^{investor}_{n_1,\infty} - O\left((F(1 - \frac{1}{n_1}))^m\right)$$

and

$$\Gamma^{investor}_{n_2,m} = \Gamma^{investor}_{n_2,\infty} - O\left((F(1 - \frac{1}{n_2}))^m\right)$$

A comparison of these two expressions does not yield a decisive result. Indeed, $\Gamma^{investor}_{n_1,\infty} > \Gamma^{investor}_{n_2,\infty}$, however, $O\left((F(1 - \frac{1}{n_1}))^m\right) > O\left((F(1 - \frac{1}{n_2}))^m\right)$. Thus for low levels of competition, it might be that a decrease in tick size will have an opposite effect to that anticipated in the perfectly competitive setting. The difference between $O\left((F(1 - \frac{1}{n_1}))^m\right)$ and $O\left((F(1 - \frac{1}{n_2}))^m\right)$ is large when m is small, when the difference between n_1 and n_2 is large, and when the right tail of F is thin. Therefore, we would expect these odd effects following a decrease in tick size in markets where: (i) the level of competition is low; (ii) the decrease in tick size is sharp; and (iii) the probability of high reservation values for dealers is small. Thus, for example, a move from a tick size of $\$\frac{1}{8}$ to a tick size of \$0.01 might have counter intuitive effects on markets with low levels of competition. Notice also that in all our examples we made use of the uniform distribution for computational convenience. Had we used instead other distributions with thinner right tails, we would get even stronger results.

7 Discussion and Conclusions

We have investigated the relation between price discreteness and competition in a dealer market. Our analysis suggests that there is an interaction between the level of competition among dealers and the level of price discreteness. In our view, this result should serve as a guidance to market designersWhen they decide on the tick size for a specific stock they should take into account the number of dealers for this specific stock. A very fine grid of prices for not highly competitive stocks enables the dealers to better exploit their market power, and extract a larger portion of the potential gains from trade.

In order to simplify the model we assumed that dealers' valuations are private and statistically independent. This assumption seems plausible given that inventory positions are the dealers' private information, and are independent across dealers. However, we claim that our main results are robust to this assumption. To see this, consider first the monopolistic dealer case. In this case, it is obvious that a high tick size restrains the monopoly, no matter what is the structure of valuations. Thus our insight about this case is intact, regardless of the specific assumption about dealers' valuations.

Consider now the perfectly competitive case. Gong and McAfee (1996) provide a convergence result that is similar to our Proposition 4, allowing for more general valuation schemes. Indeed, Gong and McAfee use continuous prices, and make heavy use of first order conditions, that we cannot use in our discrete setting. Still, the flavor of their result, showing convergence to honesty when the number of market participants becomes large should carry on to our discrete setting as well, given much weaker assumptions regarding valuations. Moreover, our results regarding the perfectly competitive market stand in line with the extant literature about discrete prices, that assumes zero profit, without modeling the process of convergence due to increasing levels of competition.

Given that dealers have opposite attitudes towards the tick size in the monopolistic and in the perfectly competitive case, it must be that their attitude changes starting from some level of competition that depends on the current tick size. Thus, low levels of competition should resemble the monopolistic case while high levels of competition should resemble the perfectly competitive case.

It should be emphasized, however, that the rate of convergence to the perfectly competitive case: $O((F(1-\frac{1}{n}))^m)$, relies heavily on the assumption of independent private valuations. Moreover, generalizing it to other valuation schemes does not seem to be an easy task. This should introduce some caution in applying it to more evolved environments.

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8 Appendix A

Proof of Proposition 1

The following definition is needed first.

Definition 1 *The function* $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ *satisfies the Milgrom-Shannon single crossing property (SCP) in* (x, θ) *if, for all* $x_H > x_L$ *and all* $\theta_H > \theta_L$*,* $h(x_H, \theta_L) \geq (>) h(x_L, \theta_L)$ *implies* $h(x_H, \theta_H) \geq (>) h(x_L, \theta_H)$*.*

Consider now our game of incomplete information with m dealers and a tick size of $\frac{1}{n}$. Let $(B_{1,...,}B_m)$ be an m-tuple of strategies. We denote by $\pi_i(b_i, y_i, B_{-i})$ the expected profit to a dealer $i \in \{1, ..., m\}$ whose value is $y_i \in [0, 1]$, when he submits a quote of $b_i \in P_n$ and the $m-1$ other dealers use the strategies $B_1, ..., B_{i-1}, B_{i+1}, ..., B_m$.

Proposition (Adapted from Athey 2001). Suppose that given any m-tuple of non-decreasing strategies $B_i : [0, 1] \rightarrow P_n$, $\pi_i(b_i, y_i, B_{-i})$ satisfies the Milgrom-Shannon single crossing property in (b_i, y_i) for all $i = 1, .., m$. Then there exists a non-decreasing pure strategy Nash equilibrium, i.e. an m-tuple $(B_1^*, ..., B_m^*)$ such that for almost all $y_i \in [0, 1]$, $B_i^*(y_i) \in \arg\max_{b \in P_n} \pi_i(b_i, y_i, B_{-i}^*)$, for all $i = 1, .., m$.

In order to accomplish our existence result we must show that Athey's result can be adapted to prove existence of symmetric equilibria, i.e. equilibria that satisfy $B_1 = B_1 = ... = B_m \equiv B$. Furthermore, we must show that our objective function satisfies the Milgrom-Shannon single crossing property given that all dealers use non-decreasing strategies. The former claim involves referring to Athey's specific proofs and notations, and is therefore deferred to Appendix B. For the latter claim, in the following lemma we show that the single crossing property is satisfied in this model no matter what strategies are used by other dealers.

Lemma 1 *For any given dealers' strategies* $(B_1, ..., B_m)$*,* $\pi_i(b_i, y_i, B_{-i})$ *satisfies the Milgrom-Shannon single crossing property for all* $i = 1, ..., m$*.*

Proof. Let $b_H, b_L \in P_n$, $y_H, y_L \in [0,1]$ and suppose $b_H > b_L$, $y_H > y_L$. Let $i \in \{1, ..., m\}$ and assume:

$$\pi_i(b_H, y_L, B^{-i}) \geq \pi_i(b_L, y_L, B^{-i}) \tag{10}$$

and suppose on the contrary that

$$\pi_i(b_H, y_H, B^{-i}) < \pi_i(b_L, y_H, B^{-i}) \tag{11}$$

We denote:

$$\Phi_i(b, B_{-i}) = \Pr \left\{ \begin{array}{l} \text{A transaction occurs, and dealer } i \text{ wins the stock} \\ \text{given that he bids } b \text{ and the other dealers use} \\ \text{the strategies } B_1, ..., B_{i-1}, B_{i+1}, ..., B_m \end{array} \right\}$$

Notice that $\Phi_i(b, B_{-i})$ is non-decreasing in b no matter what kind of strategies are used by the other $m-1$ dealers.

Equations (10) and (11) may be rephrased as follows:

$$\begin{array}{rcl} (y_L - b_H)\Phi_i(b_H, B_{-i}) & \geq & (y_{L-}b_L)\Phi_i(b_L, B_{-i}) \\ (y_H - b_H)\Phi_i(b_H, B_{-i}) & < & (y_{H-}b_L)\Phi_i(b_L, B_{-i}) \end{array}$$

But these two inequalities together imply that: $\Phi_i(b_H, B_{-i}) < \Phi_i(b_L, B_{-i})$ - a contradiction. A similar argument is used to show that the strict inequalities part of the single crossing property definition is satisfied. ■

Proof of Proposition 4

The following 4 Lemmas are required for the proof.

Lemma 2 *Let $\{a_m\}, \{b_m\}, \{c_m\}$ and $\{d_m\}$ be any convergent sequences of real numbers such that for all $m = 1, 2, 3....$: $1 \geq a_m > b_m \geq c_m > d_m \geq 0$, and denote the limits of these sequences by a, b, c and d respectively. Then the following holds:*

1. If $1 > a > b$ then $\frac{(a_m)^m - (b_m)^m}{a_m - b_m} \to 0$.
2. If $1 = a > b$ then $\frac{(a_m)^m - (b_m)^m}{a_m - b_m}$ is a bounded sequence.[17]

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[17] Notice that in this case although $a_m \to 1$, $(a_m)^m$ does not necessarily converge to 1. For example $(1 - \frac{1}{m})^m \to e^{-1}$.

3. If $b \geq c > d$ then:

$$\frac{\frac{(c_m)^m-(d_m)^m}{c_m-d_m}}{m\cdot(b_m)^{m-1}} = O(\frac{1}{m})$$

4. If $a \geq b$ then the sequence $\frac{m\cdot(b_m)^{m-1}}{\frac{(a_m)^m-(b_m)^m}{a_m-b_m}}$ is bounded from above by 1.

5. If $a \geq b \geq c > d$ then:

$$\frac{\frac{(c_m)^m-(d_m)^m}{c_m-d_m}}{\frac{(a_m)^m-(b_m)^m}{a_m-b_m}} = O(\frac{1}{m})$$

Proof. 1. Let $\varepsilon > 0$ satisfy $a < 1-\varepsilon$. For m large enough we have: $a_m < 1-\varepsilon$, and therefore: $0 \leq (a_m)^m < (1-\varepsilon)^m$. Hence by the sandwich rule $(a_m)^m \to 0$. The same argument shows that $(b_m)^m \to 0$, and since $a > b$ we are done.

2. Trivial.

3.

$$\frac{\frac{(c_m)^m-(d_m)^m}{c_m-d_m}}{m\cdot(b_m)^{m-1}} = \frac{1}{m}\cdot\frac{b_m}{c_m-d_m}\cdot\left((\frac{c_m}{b_m})^m-(\frac{d_m}{b_m})^m\right)$$

Now, $(\frac{c_m}{b_m})^m$ is bounded from above by 1, and $(\frac{d_m}{b_m})^m$ tends to zero by the argument of Part 1 of this lemma. This establishes the required result.

4. For any $m = 1, 2, ...$ the following holds:

$$\frac{m\cdot(b_m)^{m-1}}{\frac{(a_m)^m-(b_m)^m}{a_m-b_m}} = \frac{m\cdot(b_m)^{m-1}}{\sum_{i=1}^m (a_m)^{m-i}(b_m)^{i-1}} = \frac{m}{\sum_{i=1}^m (\frac{a_m}{b_m})^{m-i}}$$

Since $a_m > b_m$ each term of the sum in the denominator is greater than or equal to 1, hence the denominator is greater than m. Thus, for each m this expression is bounded from above by 1 as required.

5. For any $m = 1, 2, 3, ...$ write:

$$\frac{\frac{(c_m)^m-(d_m)^m}{c_m-d_m}}{\frac{(a_m)^m-(b_m)^m}{a_m-b_m}} = \frac{\frac{(c_m)^m-(d_m)^m}{c_m-d_m}}{m\cdot(b_m)^{m-1}}\cdot\frac{m\cdot(b_m)^{m-1}}{\frac{(a_m)^m-(b_m)^m}{a_m-b_m}}$$

The result follows now from an application of Parts 3 and 4 of this lemma. ■

Lemma 3 *Consider a market with a tick size of* $\frac{1}{n}$*. Suppose there exists a sequence of non-decreasing equilibrium strategies* $\{B_{n,m}\}_{m=1}^{\infty}$ *in markets with* m *dealers, such that* $\lim\limits_{m\to\infty} B_{n,m} = B_n$ *almost everywhere on* $[0,1]$*. Then* $\Delta_n(B_{n,m}) = \{0, \frac{1}{n}, ..., \frac{n-1}{n}\}$ *for all* m *large enough.*

Proof. Suppose on the contrary that there exists an $h \in \{0, ..., n-1\}$ and a subsequence $\{m_t\}_{t=1}^{\infty}$ such that $\frac{h}{n} \notin \Delta_n(B_{n,m_t})$ for all $t = 1, 2, ...$ Clearly[18] $h > 0$. For $k \in \{0, ..., h-1\}$ define: $Q_k \equiv \{y \in (\frac{h}{n}, \frac{h+1}{n}) : B_n(y) = \frac{k}{n}\}$. There exists a $k \in \{0, ..., h-1\}$ such that $\Pr(Q_k) > 0$. It follows that there exists a number t_0 such that for $t \geq t_0$ there exists a set of types $Q_{k,t} \subset (\frac{h}{n}, \frac{h+1}{n})$ such that $\Pr(Q_{k,t}) > 0$, and $B_{n,m_t}(y) = \frac{k}{n}$ for all $y \in Q_{k,t}$. Moreover, by choosing t_0 large enough we have that $S_k \equiv \bigcap\limits_{t=t_0}^{\infty} Q_{k,t}$ is of positive probability. Since B_{n,m_t} is an equilibrium, it follows from Equations (5) and (6) that for all $t \geq t_0$ and $y \in S_k$:

$$\frac{1}{m_t} G(\frac{k}{n})(y - \frac{k}{n}) \frac{(F(\sigma_{k+1}^{B_{n,m_t}}))^{m_t} - (F(\sigma_k^{B_{n,m_t}}))^{m_t}}{F(\sigma_{k+1}^{B_{n,m_t}}) - F(\sigma_k^{B_{n,m_t}})} \geq G(\frac{h}{n})(y - \frac{h}{n})(F(\sigma_h^{B_{n,m_t}}))^{m_t - 1}$$

Rearranging we obtain:

$$\frac{G(\frac{k}{n})(y - \frac{k}{n})}{G(\frac{h}{n})(y - \frac{h}{n})} \geq \frac{m_t (F(\sigma_h^{B_{n,m_t}}))^{m_t - 1}}{\frac{F((\sigma_{k+1}^{B_{n,m_t}}))^{m_t} - (F(\sigma_k^{B_{n,m_t}}))^{m_t}}{F(\sigma_{k+1}^{B_{n,m_t}}) - F(\sigma_k^{B_{n,m_t}})}} \qquad \text{for all } t \geq t_0$$

The l.h.s does not depend on t, however by Lemma 2 Part 3, the r.h.s tends to infinity as t increases. Thus, this inequality cannot hold for large enough t - a contradiction. ■

Lemma 4 *Consider a market with a tick size of* $\frac{1}{n}$*. Suppose there exists a sequence of non-decreasing equilibrium strategies* $\{B_{n,m}\}_{m=1}^{\infty}$ *in markets with* m *dealers, such that* $\lim\limits_{m\to\infty} B_{n,m} = B_n$ *almost everywhere on* $[0,1]$*. Then* $\Delta_n(B_n) = \{0, \frac{1}{n}, ..., \frac{n-1}{n}\}$*.*

Proof. Suppose on the contrary that there exists an $h \in \{0, 1, ..., n-1\}$ such that $\frac{h}{n} \notin \Delta_n(B_n)$. By Lemma 3 we may assume that $\frac{h}{n} \in \Delta_n(B_{n,m})$ for all $m = 1, 2, 3, ...$

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[18] Any equilibrium strategy assumes 0 with positive probability.

This implies that $\sigma_{h+1}^{B_{n,m}} > \sigma_h^{B_{n,m}}$ for all m, however $\sigma_{h+1}^{B_n} \equiv \lim_{m\to\infty} \sigma_{h+1}^{B_{n,m}} = \lim_{m\to\infty} \sigma_h^{B_{n,m}} \equiv \sigma_h^{B_n}$. We conclude that there exists a $k \in \{0, ..., h-1\}$ such that the set: $Q_k \equiv \{y \in (\frac{h}{n}, \frac{h+1}{n}) : B_n(y) = \frac{k}{n}\}$ is of positive probability. It follows that there exists a number m_0 such that for $m \geq m_0$ the sets $Q_{k,m} := \{y \in (\frac{h}{n}, \frac{h+1}{n}) : B_{n,m}(y) = \frac{k}{n}\}$ are of positive probability. Furthermore, by choosing m_0 large enough, the set $T_k \equiv \bigcap_{m=m_0}^{\infty} Q_{k,m}$ is of positive probability.

All the $B_{n,m}$ are equilibrium strategies, and by Lemma 3 both $\frac{k}{n}$ and $\frac{h}{n}$ belong to $\Delta_n(B_{n,m})$. It follows that for all $y \in T_k$ and $m \geq m_0$:

$$\frac{1}{m}G(\frac{k}{n})(y-\frac{k}{n})\frac{(F(\sigma_{k+1}^{B_{n,m}}))^m - (F(\sigma_k^{B_{n,m}}))^m}{F(\sigma_{k+1}^{B_{n,m}}) - F(\sigma_k^{B_{n,m}})} \geq \frac{1}{m}G(\frac{h}{n})(y-\frac{h}{n})\frac{(F(\sigma_{h+1}^{B_{n,m}}))^m - (F(\sigma_h^{B_{n,m}}))^m}{F(\sigma_{h+1}^{B_{n,m}}) - F(\sigma_h^{B_{n,m}})}$$

or equivalently:

$$\frac{G(\frac{k}{n})(y-\frac{k}{n})}{G(\frac{h}{n})(y-\frac{h}{n})} \geq \frac{\frac{(F(\sigma_{h+1}^{B_{n,m}}))^m - (F(\sigma_h^{B_{n,m}}))^m}{F(\sigma_{h+1}^{B_{n,m}}) - F(\sigma_h^{B_{n,m}})}}{\frac{(F(\sigma_{k+1}^{B_{n,m}}))^m - (F(\sigma_k^{B_{n,m}}))^m}{F(\sigma_{k+1}^{B_{n,m}}) - F(\sigma_k^{B_{n,m}})}}$$

The l.h.s of this inequality does not depend on m while the r.h.s tends to infinity by Lemma 2 Part 5. This yields a contradiction. ■

Lemma 5 *Consider a market with a tick size of* $\frac{1}{n}$. *Suppose there exists a sequence of non-decreasing equilibrium strategies* $\{B_{n,m}\}_{m=1}^{\infty}$ *in markets of* m *dealers, such that* $B_{n,m} \to B_n$ *almost everywhere on* $[0,1]$. *Then* $B_n = H_n$ *almost everywhere on* $[0,1]$.

Proof. In order to show that $B_n = H_n$ almost everywhere we will show that $\sigma_k^{B_n} = \lim_{m\to\infty} \sigma_k^{B_{n,m}} = \frac{k}{n}$ for all $k \in \{0, 1, ..., n-1\}$. First, notice that since each $B_{n,m}$ is an equilibrium we have $\sigma_k^{B_{n,m}} \geq \frac{k}{n}$ for all $k = 1, ..., n-1$ and $m = 1, 2, ...$ By moving to the limit we obtain: $\sigma_k^{B_n} \geq \frac{k}{n}$ for all $k = 1, ..., n-1$. Therefore, we only have to show that a strict inequality is impossible.

Let $k \in \{0, 1, ..., n-1\}$ and suppose on the contrary that $\sigma_k^{B_n} > \frac{k}{n}$. It follows that there exists a number m_0 such that for all $m \geq m_0$, $\sigma_k^{B_{n,m}} > \frac{k}{n}$. From Lemma 4 it follows that $\frac{k}{n} \in \Delta_n(B_n)$. This implies that there exists an $h \in \{k+1, ..., n\}$ such the set $Q \equiv \{y \in (\frac{h}{n}, \frac{h+1}{n}) : B_n(y) = \frac{k}{n}\}$ is of positive probability. This in turn implies that if we choose m_0 large enough the sets $Q_{k,m} \equiv \{y \in (\frac{h}{n}, \frac{h+1}{n}) : B_{n,m}(y) = \frac{k}{n}\}$ are of positive probability for all $m \geq m_0$, and furthermore the set $T_k \equiv \bigcap\limits_{m=m_0}^{\infty} Q_{k,m}$ is of positive probability. Applying Lemma 4 again, we conclude that $\frac{h}{n} \in \Delta_n(B_n)$, hence by choosing m_0 large enough we may assume that it is attained with positive probability also by $B_{n,m}$ for all $m \geq m_0$.

Since $B_{n,m}$ is an equilibrium and both $\frac{k}{n}$ and $\frac{h}{n}$ appear with positive probability, we may write for all $y \in T_k$ and $m \geq m_0$:

$$\frac{1}{m}G(\frac{k}{n})(y-\frac{k}{n})\frac{(F(\sigma_{k+1}^{B_{n,m}}))^m - (F(\sigma_k^{B_{n,m}}))^m}{F(\sigma_{k+1}^{B_{n,m}}) - F(\sigma_k^{B_{n,m}})} \geq \frac{1}{m}G(\frac{h}{n})(y-\frac{h}{n})\frac{(F(\sigma_{k+1}^{B_{n,m}}))^m - (F(\sigma_k^{B_{n,m}}))^{m_t}}{F(\sigma_{k+1}^{B_{n,m}}) - F(\sigma_k^{B_{n,m}})}$$

But Lemma 2 Part 5 implies that this cannot hold for large enough m - a contradiction. ■

We turn now to the proof of the proposition. Let $\{B_{n,m}\}$ be any sequence of equilibrium strategies in markets with m dealers. Each $B_{n,m}$ is represented by $\sigma^{B_{n,m}} \in \Sigma \subset \mathbb{R}^{n+2}$. By Lemma 5, all convergent subsequences of $\sigma^{B_{n,m}}$ converge to σ^{H_n}. Since Σ is compact this implies that $\sigma^{B_{n,m}}$ itself converges the σ^{H_n}. But this in turn implies that $B_{n,m}$ converges to H_n almost everywhere. ■

Proof of Proposition 5

We prove the second part of the proposition first.

By Proposition 4, there exists a number $M(n)$ such that for all $m \geq M(n)$ we have $\Delta_n(B_{n,m}) = \{0, \frac{1}{n}, ..., \frac{n-1}{n}\}$. Let $m \geq M(n)$. For any $h \in \{1, ..., n-1\}$,$k \in \{0, ..., h-1\}$ and $y = \sigma_h^{B_{n,m}}$ we have:

$$G(\frac{h-1}{n})(\sigma_h^{B_{n,m}}-\frac{h-1}{n})\frac{(F(\sigma_h^{B_{n,m}}))^m - (F(\sigma_{h-1}^{B_{n,m}}))^m}{F(\sigma_h^{B_{n,m}}) - F(\sigma_{h-1}^{B_{n,m}})} = G(\frac{h}{n})(\sigma_h^{B_{n,m}}-\frac{h}{n})\frac{(F(\sigma_{h+1}^{B_{m,m}}))^m - (F(\sigma_h^{B_{n,m}}))}{F(\sigma_{h+1}^{B_{n,m}}) - F(\sigma_h^{B_{n,m}})}$$

Solving for $\sigma_h^{B_{n,m}}$ yields:

$$\sigma_h^{B_{n,m}} = \frac{h}{n} + \frac{1}{n} \cdot \frac{1}{\frac{G(\frac{h}{n})}{G(\frac{h-1}{n})} \cdot \frac{F(\sigma_h^{B_{n,m}})-F(\sigma_{h-1}^{B_{n,m}})}{F(\sigma_{h+1}^{B_{n,m}})-F(\sigma_h^{B_{n,m}})} \cdot \frac{(F(\sigma_{h+1}^{B_{n,m}}))^m-(F(\sigma_h^{B_{n,m}}))^m}{(F(\sigma_h^{B_{n,m}}))^m-(F(\sigma_{h-1}^{B_{n,m}}))^m} - 1}$$

By Lemma 2 Part 5 we obtain that: $\sigma_h^{B_{n,m}} = \frac{h}{n} + \frac{1}{n} \cdot O(\frac{1}{m})$ for all $h \in \{0, ..., n-1\}$ as required.

As for the first part of the proposition: the probability of getting a dishonest quote by a dealer is: $\sum_{h=0}^{n-1}(\sigma_h^{B_m} - \frac{h}{n})$. The previous result immediately implies that this probability is $O(\frac{1}{m})$. ■

Proof of Proposition 6

Part 1. From Equation (7) we have:

$$\Gamma_{n,m}^{investor}(B_{n,m}) = \sum_{k=1}^{n-1} \gamma(\frac{k}{n}) \left[(F(\sigma_{k+1}^{B_{n,m}}))^m - (F(\sigma_k^{B_{n,m}}))^m\right] \tag{12}$$

By Proposition 4, for all $k = 1, ..., n-1$, $\sigma_k^{B_{n,m}} \to \frac{k}{n} < 1$. Hence, by an argument similar to Lemma 2 Part 1, $(F(\sigma_k^{B_{n,m}}))^m \to 0$. It follows that as m tends to infinity we may neglect all the terms of the summation except $k = n-1$, since they all tend to zero. On the other hand, $F(\sigma_n^{B_{n,m}}) = 1$ for all m. Thus,

$$\Gamma_{n,\infty}^{investor} = \lim_{m\to\infty} \gamma(\frac{n-1}{n}) \left[1 - (F(\sigma_{n-1}^{B_{n,m}}))^m\right] = \gamma(1 - \frac{1}{n}) \tag{13}$$

By using the Taylor expansion for $\gamma(\cdot)$ and applying integration by parts we obtain:

$$\gamma(1 - \frac{1}{n}) = \gamma(1) - \frac{1}{n} + O(\frac{1}{n^2}) = 1 - \mu_G - \frac{1}{n} + O(\frac{1}{n^2})$$

as required.

Part 2.

From Equation (8) we have:

$$\Gamma_{n,m}^{dealers}(B_{n,m}) = \sum_{k=1}^{n-1} G(\frac{k}{n}) \frac{(F(\sigma_{k+1}^{B_{n,m}}))^m - (F(\sigma_k^{B_{n,m}}))^m}{F(\sigma_{k+1}^{B_{n,m}}) - F(\sigma_k^{B_{n,m}})} \int_{y=\sigma_k^{B_{n,m}}}^{\sigma_{k+1}^{B_{n,m}}} (y - \frac{k}{n}) f(y) dy$$

By the same argument used in Part 1 of this proposition, all the terms in the summation except $k = n-1$ should be neglected as m tends to infinity.

It follows that:

$$\Gamma_{n,\infty}^{dealers} = \lim_{m\to\infty} G(\frac{n-1}{n}) \frac{1 - (F(\sigma_{n-1}^{B_{n,m}}))^m}{1 - F(\sigma_{n-1}^{B_{n,m}})} \int_{y=\sigma_{n-1}^{B_{n,m}}}^{1} (y - \frac{n-1}{n}) f(y) dy$$

Denote: $\varphi(t) \equiv \int_0^t F(y)dy$. By Proposition 4, $\sigma_{n-1}^{B_{n,m}} \rightarrow \frac{n-1}{n}$. Hence, from the continuity of F and by applying integration by parts we obtain:

$$\Gamma_{n,\infty}^{dealers} = \frac{G(1-\frac{1}{n})}{1-F(1-\frac{1}{n})}\left[\frac{1}{n} - \varphi(1) + \varphi(1-\frac{1}{n})\right]$$

Using the Taylor expansion yields:

$$\begin{aligned} \varphi(1-\frac{1}{n}) &= \varphi(1) - \frac{1}{n} + \frac{f(1)}{2n^2} + O(\frac{1}{n^3}) \\ F(1-\frac{1}{n}) &= 1 - \frac{f(1)}{n} + O(\frac{1}{n^2}) \\ G(1-\frac{1}{n}) &= 1 - O(\frac{1}{n}) \end{aligned}$$

Therefore,

$$\Gamma_{n,\infty}^{dealers}(B_m) = (1-O(\frac{1}{n}))\frac{\frac{f(1)}{2n^2} + O(\frac{1}{n^3})}{\frac{f(1)}{n} - O(\frac{1}{n^2})} = \frac{1}{2n} + O(\frac{1}{n^2})$$

Part 3. Follows from adding up the results of Part 1 and Part 2. ■

Proof of Proposition 7

Part 1. Let $m \geq 1$. We assume below that $n \geq 3$.[19] From Equations (12) and (13) we have:

$$\begin{aligned} \Gamma_{n,\infty}^{investor} - \Gamma_{n,m}^{investor} &= \gamma(1-\frac{1}{n}) - \sum_{k=1}^{n-1}\gamma(\frac{k}{n})\left[(F(\sigma_{k+1}^{B_{n,m}}))^m - (F(\sigma_k^{B_{n,m}}))^m\right] \\ &= \left[\gamma(1-\frac{1}{n}) - \gamma(1-\frac{2}{n})\right](F(\sigma_{n-1}^{B_{n,m}}))^m + \gamma(1-\frac{2}{n})(F(\sigma_{n-2}^{B_{n,m}}))^m \\ &\quad -\sum_{k=1}^{n-3}\gamma(\frac{k}{n})\left[(F(\sigma_{k+1}^{B_{n,m}}))^m - (F(\sigma_k^{B_{n,m}}))^m\right] \end{aligned}$$

Therefore:

$$\begin{aligned} \frac{\Gamma_{n,\infty}^{investor} - \Gamma_{n,m}^{investor}}{(F(1-\frac{1}{n}))^m} &= \left[\gamma(1-\frac{1}{n}) - \gamma(1-\frac{2}{n})\right]\left(\frac{F(\sigma_{n-1}^{B_{n,m}})}{F(1-\frac{1}{n})}\right)^m \\ &\quad +\gamma(1-\frac{2}{n})\left(\frac{F(\sigma_{n-2}^{B_{n,m}})}{F(1-\frac{1}{n})}\right)^m - \frac{\sum_{k=1}^{n-3}\gamma(\frac{k}{n})\left[(F(\sigma_{k+1}^{B_{n,m}}))^m - (F(\sigma_k^{B_{n,m}}))^m\right]}{(F(1-\frac{1}{n}))^m} \end{aligned}$$

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[19] The proof is the same for $n < 3$ with small modifications.

By Proposition 4, as m tends to infinity $F(\sigma_k^{B_{n,m}}) \to F(\frac{k}{n})$ for all $k = 1, ...n-1$. This immediately implies that the second and the third terms tend to zero. Let us denote the first term by A_1. By Proposition 5 we have:

$$A_1 = \left[\gamma(1-\frac{1}{n}) - \gamma(1-\frac{2}{n})\right] \left(\frac{F(1-\frac{1}{n}+\frac{1}{n}O(\frac{1}{m}))}{F(1-\frac{1}{n})}\right)^m$$

As m tends to infinity the term in parentheses tends to $\exp(\frac{1}{n(1-\frac{1}{n})})$, which is of course finite. Thus, we have shown that $\frac{\Gamma^{investor}_{n,\infty} - \Gamma^{investor}_{n,m}}{(F(1-\frac{1}{n}))^m}$ tends to a finite limit as m tends to infinity, as required.

Similar arguments are used to show Part 2. Part 3 follows from Parts 1 and 2. ■

9 Appendix B

In this appendix we show how the result of Athey (2001) should be modified to achieve existence of a symmetric equilibrium. We use her notations and results freely. Specifically, all theorems, lemmas, definitions and equations in this appendix pertain to her paper.

Definition: We say that a game of incomplete information is symmetric if:

1. The supports of types are identical: $T_1 = T_2 = ... = T_I \equiv T = [\underline{t}, \bar{t}]$.
2. The actions sets are identical: $\mathcal{A}_1 = \mathcal{A}_2 = ... = \mathcal{A}_I \equiv \mathcal{A}$
3. The distribution of types is exchangeable: for any permutation $\pi : \{1, .., I\} \to \{1, ..., I\}$, the joint density function $f(\cdot)$ satisfies: $f(t_1, ..., t_I) = f(t_{\pi(1)}, ...t_{\pi(I)})$.
4. The payoffs of players are symmetric: for any permutation $\pi : \{1, .., I\} \to \{1, ..., I\}$, and player $i \in \{1, ..., I\}$, $u_i(a_1, ..., a_I, t_1, ..., t_I) = u_{\pi(i)}(a_{\pi(1)}, ..., a_{\pi(I)}, t_{\pi(1)}, ..., t_{\pi(I)})$.

In order to prove existence of a symmetric equilibrium we replicate Athey's proof step by step, however, the symmetry of the game allows us to use individual players instead of I-tuples as in the original proof. Eventually, this will enable us to use Kakutani's fixed point theorem on a correspondence pertaining to a specific player. Since players are symmetric this results in a symmetric equilibrium. This method of proving existence of symmetric equilibria is based on Nash's (1951) classic paper.

We assume that the actions set is finite. Let $\mathcal{A} = \{A_0, A_1, ..., A_M\}$ be the set of potential actions in ascending order, where $M + 1$ is the number of potential actions. Define: $\Sigma \equiv \{x \in T^{M+2} : x_0 = \underline{t} \leq x_1 \leq x_2 \leq ... \leq x_M \leq x_{M+1} = \bar{t}\}$.[20] A non-decreasing strategy is a non-decreasing mapping $\alpha : T \to \mathcal{A}$. Any non-decreasing strategy α may be represented by a vector $x \in \Sigma$ using Definition 4.

Let us consider a specific player. Given $x \in \Sigma$, let $V(a, t, x)$ denote the expected payoff to our specific player, given that his type is $t \in T$, he chooses an action $a \in \mathcal{A}$

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[20] Notice that here Σ is a set in $\mathbb{R}^{M+2}$ while in Athey (2001) it is a set in $\mathbb{R}^{I(M+2)}$.

and the other $I-1$ players use strategies that are consistent with x (they all use identical strategies). The expression for $V(a,t,x)$ is similar to Equation (1), only that all the other $I-1$ players use the same strategy (which is consistent with x). $V(a,t,x)$ satisfies the SCP-IR in $(a;t)$ for all $x \in \Sigma$. Denote: $a^{BR}(t|x) = \arg\max_{a \in A} V(a,t,x)$. Now define:

$$\Gamma(x) = \{y \in \Sigma : \exists \alpha(\cdot) \text{ which is consistent with } y \text{ s.t. } \forall t \in T,\ \alpha(t) \in a^{BR}(t|x)\}$$

$\Gamma(\cdot)$ is a correspondence taking points in $\Sigma \subset \mathbb{R}^{M+2}$ into subsets of Σ.[21] Exactly the same arguments of Athey (2001) are used to show that $\Gamma(\cdot)$ satisfies the conditions of Kakutani's fixed point theorem. Let $x \in \Sigma$ be a fixed point, and let $\beta(\cdot)$ be a non-decreasing strategy consistent with x. By construction $\beta(\cdot)$ is a best response for a player when the other $I-1$ players use $\beta(\cdot)$. Therefore, $(\beta(\cdot), ..., \beta(\cdot))$ is a symmetric equilibrium. Thus, we have proved the following variant of Theorem 1:

Theorem 1': Consider a symmetric game of incomplete information. Assume A1 and the SCC hold. If $\mathcal{A}$ is finite, this game has a symmetric, non-decreasing, pure strategy Nash Equilibrium.

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[21] Compare to Athey (2001), where Γ is a correpondence from $\Sigma \subset R^{I(M+2)}$ to itself.

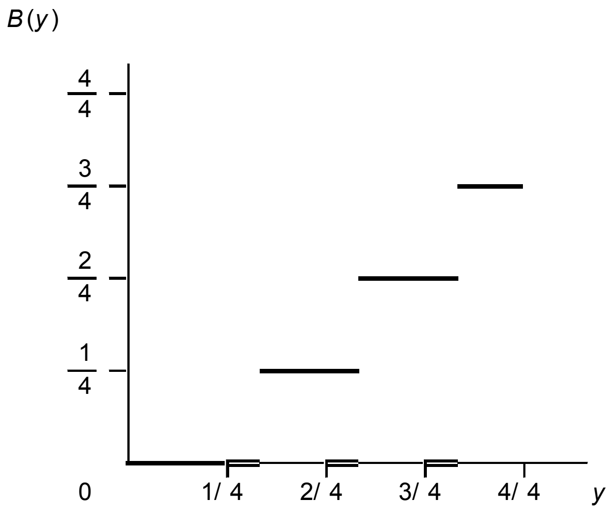

Figure 1: Equilibrium with a limited level of competition