Motivating entrepreneurial activity in a firm*

Antonio E. Bernardo[†]

Hongbin Cai[‡]

Jiang Luo[§]

April 2004

*Preliminary and incomplete.

[†]Corresponding author: UCLA Anderson School, Los Angeles, CA 90095-1481, USA. ph: (310) 825-2198. e-mail: abernard@anderson.ucla.edu

[‡]Department of Economics, UCLA, Los Angeles, CA 90095-1477, USA

[§]Department of Finance, HKUST, Clear Water Bay, Hong Kong

Abstract

We examine a model in which a firm must motivate a manager to engage in entrepreneurial effort in order to improve the investment opportunities it faces. Such effort can be interpreted as an activity that enhances the average quality of investments available to the firm (e.g., search) or improves the expected performance of a given investment *prior* to its funding (e.g., product innovation). We show that the firm motivates entrepreneurial activity by overinvesting capital (relative to first-best) and increasing the strength of performance-based incentives in high-quality projects. We also show that, unlike the standard agency model, the strength of performance-based incentives and uncertainty may be positively related.

1 Introduction

Fostering entrepreneurial activity within a firm is critical to its success, especially in high-growth industries. Many observers have argued, for example, that in rapidly changing industries large, diversified firms are at a disadvantage to smaller, more focused firms because of structural and informational impediments to entrepreneurial activity such as slow-moving and bureaucratic organizations (Henderson, 1993), an over-reliance on the needs of existing clients (Christensen, 1997), internal rent-seeking behavior which leads to a "socialistic" allocation of capital across divisions (Rajan, Servaes, and Zingales, 2000), and difficulties assessing the quality of growth opportunities outside their core business (e.g., Stein, 1997; Scharfstein and Stein, 2000). In this paper, we argue that a firm must tap its workers' abilities to gather information and use their many talents to improve and expand its growth opportunities (i.e., "entrepreneurial effort"). For example, salespeople can suggest product innovations based on information they gather about client needs, division managers can search for new markets, or engineers can expend effort to develop new products. One way to encourage such privately costly activity in a firm is to link pay to investment performance.[1]. But unless the firm can commit to funding promising new ideas and ventures, workers will recognize that any entrepreneurial activity will be wasted and may decide to take their ideas elsewhere.[2]

We examine these issues by considering a risk-neutral firm that must motivate a risk-neutral manager to engage in "entrepreneurial effort" to improve its investment opportunities. Such effort can be interpreted as an activity which enhances the aver-

[1]This may also help to *attract* creative, risk-taking individuals to the firm, however, we do not formally model this selection process (see, e.g., Jovanovic, 1979)

[2]The classic example of this is Xerox whose executives resisted the commercialization of numerous valuable technologies, such as the ethernet and the graphical user interface, developed at their Palo Alto Research Center (PARC). Most of the value from these technologies was captured by workers who left Xerox to start new companies. Gompers, Lerner, and Scharfstein (2004) provides a detailed empirical analysis of the forces leading to "entrepreneurial spawning" in firms.

age quality of investments available to the firm (e.g., search) or improves the expected performance of a given investment *prior* to its funding (e.g., product innovation). An important feature of our model is that after expending entrepreneurial effort, the manager also learns private information which is unknown to the firm. Consequently, the firm must decide how much capital to allocate to the project based on a report prepared by the better-informed manager. Once capital is allocated, the manager can also enhance the value of the project by exerting "managerial effort" representing activities that can enhance project performance *after* it is funded. The manager's entrepreneurial and managerial efforts are both assumed to be non-verifiable, non-contractible, and privately costly. In this setting, we examine how the firm can use two instruments, a capital allocation rule and the manager's compensation package, to provide incentives for the manager to exert both entrepreneurial and managerial effort, and to report project quality truthfully.

We demonstrate that the firm motivates entrepreneurial activity by overinvesting (relative to first-best) and strengthening performance-based incentives in high-quality projects. Since capital and managerial effort are assumed to be complementary, the manager also devotes too much managerial effort to high-quality projects. The overinvestment result depends critically on the firm's need to motivate entrepreneurial activity. If the firm need not motivate such activity, the optimal mechanism simply trades off the benefits of motivating managerial effort against the costs of encouraging truthful reporting. The firm can motivate managerial effort by strengthening performance-based incentives and increasing the capital allocation; however, any commitment to increase the expected compensation for a given realization of project quality must also include a commitment to offer at least as much expected compensation to the manager for all higher realizations of project quality otherwise she will not report truthfully. In other words, the total costs of providing incentives includes the indirect cost of increasing the compensation of the manager for all higher realizations of project quality. This makes the costs of strengthening performance-based incentives and capital excessive, especially for low-quality projects, and the optimal mechanism without entrepreneurial effort results in the underinvestment of capital (and managerial effort). The novel feature of our paper is that the firm also wishes to motivate entrepreneurial effort which can be achieved by further strengthening incentives and allocating more capital over some range of project qualities; however, since the indirect cost of doing so is smaller for the higher quality projects, the firm optimally overinvests (underinvests) capital and the manager provides too much (too little) managerial effort for high (low) quality projects. Moreover, because the costs of motivating entrepreneurial effort are higher than in the first-best allocation, the manager provides too little entrepreneurial effort.

The optimal second-best allocation can be implemented by a linear managerial compensation contract - salary plus a share of the project cash flows. We show that, unlike the standard principal-agent model, there may be a positive relation between uncertainty and the strength of performance incentives. In our model, more uncertainty increases the value of entrepreneurial effort because it is more likely that the manager will uncover a very good (or very bad) project and the firm has the option not to invest in bad projects. The firm can motivate greater entrepreneurial effort by offering expected compensation which increases in such effort. Once again, however, the firm finds it relatively inexpensive to do this by strenghtening performance incentives for high quality projects.

This paper is related to the literature examining the role of incentives in the capital budgeting process when there is asymmetric information and/or managerial moral hazard. This literature typically shows that capital should be rationed in some states to reduce managerial information rents (see, e.g., Antle and Eppen, 1985). We focus on the role of managerial compensation contracts in mitigating information and incentive problems (see also Bernardo, Cai, and Luo, 2001; Garcia, 2003); however, firms may also rely on other incentive mechanisms such as ex post auditing (e.g., Harris and Raviv, 1996), the external managerial labor market (e.g., Holmstrom and Ricart i Costa, 1986), and managerial reputation within the firm (e.g., Ozbas, 2004). These alternative mechanisms are sensible in some settings but not in others. For example, ex post auditing may be neither feasible (e.g., when the private information is sufficiently technical) nor desirable (e.g., the firm wants to keep the information from competitors). It may also be difficult for a manager to build a reputation for truthfulness when projects have a long gestation period (e.g., drug R&D). The extant literature also assumes that the principal has no control over the investment opportunity set.[3] This assumption is critical. For example, in Harris and Raviv (1996) the consideration of auditing costs leads the firm to overinvest (underinvest) in low (high) quality projects while in Bernardo, Cai, and Luo (2001) firms always underinvest in the presence of asymmetric information and managerial moral hazard. We show, however, that the firm optimally overinvests in high-quality projects to motivate the manager to give entrepreneurial effort prior to project funding.

This paper also contributes to the extensive literature on incentive provision in the presence of managerial moral hazard. The standard model in this literature considers the situation in which a risk-neutral firm must provide incentives to a risk-averse manager to expend value-enhancing, but privately costly, effort (e.g., Holmstrom, 1979; Rosen, 1982; Holmstrom and Milgrom, 1987). In the standard model, the impact of managerial effort on firm value is assumed to be additive and non-stochastic and the only uncertainty is measurement error. Consequently, the marginal benefit of providing high-powered incentives is independent of uncertainty but the cost of providing high-powered incentives to a risk-averse manager is high when uncertainty (i.e., measurement error) is high; thus, a key empirical prediction of the standard model is that risk and incentives are negatively related. The empirical evidence, however, suggests a positive

[3]An interesting exception is Berkovitch and Israel (2004) in which managers costlessly learn about a fixed set of projects but can choose which projects to propose to the firm. They show that a ratio-based investment criterion such as the IRR can be more efficient than the NPV criterion. They do not consider managerial moral hazard nor do they fully characterize the firm's optimal investment policy and incentive contracts.

relation between uncertainty and incentives (Prendergast, 2002). In our model, more uncertainty increases the value of entrepreneurial effort and thus the marginal benefit of providing performance-based incentives; however, more uncertainty decreases the marginal costs of providing performance-based incentives for high-quality projects, thus leading to the positive relation between uncertainty and incentives for the upper range of project qualities.[4]

2 The model

We consider a risk-neutral firm with unlimited access to capital to invest.[5] The firm can hire a risk-neutral manager to expend entrepreneurial effort, $z \geq 0$, to improve the average quality of projects in the investment opportunity set. Specifically, we assume that the firm selects a project with quality, denoted t, drawn from a normal distribution with mean z and variance σ^2, i.e., $t \sim N(z, \sigma^2)$.[6] One interpretation of this modeling device is that the firm has access to a single investment project and the manager can engage in product innovation or development to improve the average quality of the project. Another interpretation is that the manager engages in search activity to improve the average quality of a set of mutually exclusive investment opportunities from which

[4]Prendergast (2002) makes a similar prediction in a model in which firms delegate more authority to managers in uncertain environments but constrain managerial discretion by linking pay to performance. Shi (2003) also shows a positive relation between uncertainty and incentives but in a model where the manager gives effort to learn about a given investment rather than improve the investment opportunity set.

[5]We assume the firm has access to capital but the manager does not, so the firm is indispensable to the production process. Gertner, Scharfstein, and Stein (1994) and Stein (1997) explicitly model the productive role of headquarters to help understand the costs and benefits of internal and external capital markets.

[6]Szalay (2003) uses a similar modeling device in the context of a procurement problem where the principal's effort can determine the distribution of types.

the firm chooses to invest in a single project. In this interpretation, the parameter σ^2 represents the dispersion of potential project qualities and is a measure of the uncertainty of the economic environment. Prior to its funding, the manager learns the project quality t precisely; however, the firm does not observe the project quality and must decide how much capital to allocate based on the manager's report of project quality, denoted $\hat{t}$. Once the project is funded, the manager can also enhance its cash flows by exerting managerial effort, $x \geq 0$. Thus, a key distinction between the two types of efforts is that entrepreneurial effort represents activities undertaken prior to project financing while managerial effort represents activities undertaken after project financing. The entrepreneurial effort, z, and managerial effort, x, are assumed to be privately costly; specifically, we assume the manager's private cost of entrepreneurial effort is given by $g(z) \equiv 0.5\gamma z^2$, and the private cost of managerial effort is given by $g(x) \equiv 0.5\gamma x^2$, where $\gamma \geq 0$ is the manager's effort-aversion parameter.[7]

For tractability, we assume the project generates cash flows according to the specification:

$$V = (\delta t + \theta x)k - 0.5k^2 + \epsilon,$$

where $k \geq 0$ is the capital allocated by the firm, $x \geq 0$ is the manager's managerial effort, and ϵ is a mean-zero random variable. The parameters $\delta \geq 0$ and $\theta \geq 0$ measure the importance of project quality and the manager's managerial effort to project cash flows, respectively. The cash flow specification, V, has many intuitive features; for example, the marginal product of capital is increasing in project quality and the level of managerial effort.[8]

[7]Laffont and Tirole (1986, 1993) also examine the fundamental tradeoff between moral hazard and asymmetric information in the context of regulating a single monopoly with unobserved efficiency and non-contractible "managerial" effort. Our model also considers entrepreneurial effort and is concerned with a very different problem but uses many similar solution techniques.

[8]Since everyone is risk-neutral in our model, the mean zero noise term and its distribution have no

We assume the firm can use two instruments, a capital allocation rule and the manager's compensation package, to provide incentives for the manager to exert appropriate effort (both entrepreneurial and managerial) and to report truthfully about project quality. We assume the manager can leave the firm at any time and obtain the reservation utility, $\bar{U}$, from outside opportunities. The firm's problem is then to maximize expected project cash flows net of compensation costs by choosing a capital allocation rule, $k(\hat{t})$, depending on the manager's report about project quality $\hat{t}$; and managerial compensation, $w(\hat{t}, V)$, depending on both the report and the project outcome V. Importantly, we assume that the project quality t is unobservable and non-verifiable by the firm ex post; therefore, contracts cannot be written on t directly. Moreover, the manager's entrepreneurial and managerial efforts are assumed to be unobservable and non-verifiable by the firm; therefore, the contracts cannot be written on z or x. Finally, the firm uses the correct distribution to calculate expected payoffs, i.e., the firm designs a mechanism to motivate entrepreneurial effort of z then calculates the expected payoff to the project assuming that the manager does in fact provide effort z.

To summarize, the sequence of moves of the game is as follows:

date 0: The firm offers the manager a compensation package and capital allocation rule $\{w(\hat{t}, V), k(\hat{t})\}$.

date 1: The manager expends entrepreneurial effort, z, and observes t from $N(z, \sigma^2)$.

date 2: The manager chooses whether or not to accept the contract. If she accepts, she reports project quality $\hat{t}$.

date 3: The firm allocates capital of $k(\hat{t})$ and the manager chooses managerial effort x.

date 4: The project cash flows are realized and distributed to shareholders less the compensation, $w(\hat{t}, V)$, paid to the manager.

effect on our results but is introduced to demonstrate their generality (Laffont and Tirole, 1986).

As is standard in revelation games, we assume the firm can commit to the capital allocation rule offered to the manager at date 0; otherwise, the firm would optimally choose a capital allocation at date 3 that is different than the capital promised, $k(\hat{t})$, at date 0. If, however, the manager knew this she would not report truthfully at date 2. One natural argument in defence of this assumption is that the firm's commitment is the result of (unmodelled) reputational concerns when it must play this investment game often in the future. Finally, note that we make the important assumption that the manager can reject the contract offered at date 0 after observing the project quality. This assumption ensures the sub-optimality of selling the firm to the risk-neutral manager at date 0.

3 Equilibria

For all that follows, we make the following parameter assumptions:

(A1) $\gamma > (\delta^2 + \theta^2)(1 + \dfrac{\delta^2}{\theta^2})$.

(A2) $1.5\theta^2 > \gamma$.

Assumption (A1) requires that the marginal cost of managerial effort is increasing relatively fast compared to its marginal benefit so that the objective function of the optimal mechanism design program is concave. Assumption (A2) requires that managerial effort is sufficiently important to the productivity of capital so that entrepreneurial effort is positive in the optimal mechanism.

3.1 First-best solution

Let $\Phi(.)$ (and $\phi(.)$) denote the c.d.f. (and p.d.f.) of the standard normal distribution and $\mu(.) \equiv (1-\Phi(.))/\phi(.)$ denote the inverse of its hazard rate.[9] The first-best solution maximizes the expected total surplus:

$$
\begin{aligned}
\max_{z,x(t),k(t)\geq 0} \quad & \int_{-\infty}^{\infty}\Big[E_\epsilon V - 0.5\gamma x^2\Big] d\Phi(\frac{t-z}{\sigma}) - 0.5\gamma z^2 \\
= \quad & \int_{-\infty}^{\infty}\Big[(\delta t + \theta x)k - 0.5k^2 - 0.5\gamma x^2\Big] d\Phi(\frac{t-z}{\sigma}) - 0.5\gamma z^2,
\end{aligned}
$$

Proposition 1. *The first-best entrepreneurial effort z^{fb} is determined by the following first-order condition:*

$$
0 = \frac{\delta^2}{\gamma}\Big[\phi(\frac{z^{fb}}{\sigma}) + \frac{z^{fb}}{\sigma}\Phi(\frac{z^{fb}}{\sigma})\Big] - (1-\frac{\theta^2}{\gamma})\frac{z^{fb}}{\sigma}.
$$

The first-best managerial effort and capital allocation are given by:

$$
\begin{aligned}
x^{fb}(t) &= \frac{\theta\delta}{\gamma-\theta^2}\max(0,t), \\
k^{fb}(t) &= \frac{\gamma\delta}{\gamma-\theta^2}\max(0,t).
\end{aligned}
$$

The first-best capital allocation $k^{fb}(t)$ and managerial effort $x^{fb}(t)$ increase in t (project quality). The first-best entrepreneurial effort z^{fb}, capital allocation $k^{fb}(t)$, and managerial effort $x^{fb}(t)$ increase in δ (the importance of project quality) and θ (the importance of managerial effort), and decreases in γ (effort-aversion). Entrepreneurial effort z^{fb} also increases in σ (the dispersion of potential project quality).

[9] Below we shall use two important properties of the standard normal distribution: $\mu' < 0$ and $\mu'' > 0$.

Proof: See the Appendix.

In the first-best outcome, the marginal product of capital increases in t and δ; therefore, the first-best capital allocations increase in t and δ. Since capital and managerial effort are complementary this implies that first-best managerial effort also increases in t and δ. Conversely, the marginal benefit (cost) of managerial effort increases in θ (γ); therefore, the first-best managerial effort (and capital allocation because of complementarities) increases (decreases) in θ (γ). Finally, the first-best entrepreneurial effort increases in the uncertainty parameter σ. The reason is that entrepreneurial effort is more likely to uncover a very good (or very bad) investment opportunity when the dispersion of potential project qualities is greater; therefore, since the firm always has the option not to invest the payoff to increasing entrepreneurial effort increases in σ^2.

3.2 Optimal second-best allocation

We now consider the firm's problem when the manager's efforts and information are unobservable and non-contractible. By the Revelation Principle we can, without loss of generality, restrict our attention to direct revelation mechanisms in which the manager reports the project quality truthfully. Let $U(t,\hat{t},x) \equiv E_\epsilon w(\hat{t},V) - 0.5\gamma x^2$ denote the expected utility (expectation over ϵ) of the manager when she observes t, reports $\hat{t}$, and exerts managerial effort x (note this assumes the disutility of entrepreneurial effort is sunk). We can then write the firm's optimization problem as follows:

$$\max_{z,w(t,V),x(t),k(t)} E\Pi \equiv \int_{-\infty}^{\infty} E_\epsilon(V-w)d\Phi(\frac{t-z}{\sigma})$$

subject to

(i) $x(t,\hat{t}) \in \arg\max_x U(t,\hat{t},x)$, (IC1)

(ii) $t \in \arg\max_{\hat{t}} U(t,\hat{t},x(t,\hat{t}))$, (IC2)

$$\text{(iii)} \quad z \in \arg\max_{\hat{z}} \int_{-\infty}^{\infty} U(t,t,x(t,t))d\Phi(\frac{t-\hat{z}}{\sigma}) - 0.5\gamma\hat{z}^2, \tag{IC3}$$

$$\text{(iv)} \quad U(t,t,x(t,t)) \geq \bar{U} \ \forall t, \tag{IR}$$

$$\text{(v)} \quad z, x(t), k(t) \geq 0. \tag{NN}$$

The first incentive compatibility constraint (IC1) states that the manager chooses the level of managerial effort optimally given the true project quality and her report about project quality (which determines the capital allocation). The second incentive compatibility constraint (IC2) restricts our attention to direct revelation mechanisms in which it is optimal for the manager to tell the truth about project quality. The third incentive compatibility constraint (IC3) states that the manager chooses entrepreneurial effort to maximize her net expected payoff, which equals her expected payoff from playing the direct revelation mechanism minus the disutility of entrepreneurial effort. The individual rationality constraint (IR) requires that after knowing the realization of project quality at date 1 the manager must achieve expected utility at least as high as her outside option by playing the direct revelation mechanism (entrepreneurial effort cost is already sunk at this stage of the game hence it does not affect her participation decision at this time).[10] The non-negativity constraint (NN) requires that entrepreneurial and managerial efforts and capital allocation are non-negative. Note that here we employ the notation $x(t) \equiv x(t,t)$ for simplicity.

For our main proposition we must consider the following equation system:

[10]The date-0 participation constraint requiring the manager to achieve expected utility at least as great as $\bar{U}$ *prior* to exerting entrepreneurial effort is implied by the other constraints in equilibrium. Under the constraints (IC1), (IC2), and (IR), we show below that the manager's expected payoff from playing the direct revelation mechanism is the sum of $\bar{U}$ and her expected information rent which is always positive even when she chooses entrepreneurial effort $z = 0$. In other words, the manager can get at least $\bar{U}$ by exerting zero entrepreneurial effort, therefore, with optimal entrepreneurial effort the manager's expected net payoff must be greater than $\bar{U}$ and thus the date-0 participation constraint is satisfied.

$$\lambda_n^* = \frac{\delta^2}{\gamma}\Big[\phi(z_n^*) + z_n^*\Phi(z_n^*)\Big] + \frac{\theta^2}{\gamma}z_n^* \tag{1}$$

$$z_n^* = \frac{\delta^2}{\theta^2}\left[\frac{\gamma}{\gamma-\theta^2}(t_n^* + \lambda_n^*)\Phi(z_n^* - t_n^*) - \Big[\phi(z_n^* - t_n^*) + z_n^*\Phi(z_n^* - t_n^*)\Big]\right] \tag{2}$$

$$t_n^* = \frac{\gamma}{\theta^2}\left[\mu(t_n^* - z_n^*) - \lambda_n^*\right] \tag{3}$$

In the Appendix, we prove that these equations have a unique solution $(\lambda_n^*, z_n^*, t_n^*)$, where $\lambda_n^*, z_n^*, t_n^* > 0$. Clearly $\lambda_n^*, z_n^*, t_n^*$ are all independent of σ (the subscript n stands for "normalization" by σ).

Proposition 2. *The optimal second-best allocation is characterized as follows:*

(i) The entrepreneurial effort is $z^* = \sigma z_n^*$.

(ii) If $t < 0$*, then* $x^*(t) = k^*(t) = 0$.

(iii) If $0 \le t < t^* = \sigma t_n^*$*, then* $x^*(t) = 0$ *and* $k^*(t) = \delta t$.

(iv) If $t \ge t^*$*, then*

$$x^*(t) = \frac{\theta\delta}{\gamma-\theta^2}\Big[t + \frac{\sigma\gamma}{\theta^2}(\lambda_n^* - \mu(t/\sigma - z_n^*))\Big], \tag{4}$$

$$k^*(t) = \frac{\gamma\delta}{\gamma-\theta^2}\Big[t + \sigma(\lambda_n^* - \mu(t/\sigma - z_n^*))\Big]. \tag{5}$$

Proof: See the Appendix.

As in the first-best solution, the capital allocation $k^*(t)$ and managerial effort $x^*(t)$ are non-decreasing in the project quality t. If the manager reports that the project is low quality, $t < 0$, the firm allocates no capital. If the manager reports that the project is intermediate quality, $0 \le t < t^*$, the firm funds the project but does not provide incentives (recommend) for the manager to exert managerial effort. If the manager reports that the project is high quality, $t \ge t^*$, the firm funds the project and also provides incentives (recommends) for the manager to exert managerial effort to increase the project's expected cash flows.

The following corollary compares the optimal second-best allocation to the first-best allocation.

Corollary 1. [Comparison with first-best]

(i) The optimal capital allocation and managerial effort are lower than the first-best for low project qualities and are higher than the first-best for high project qualities, i.e., there exists $\bar{t} > t^*$ such that $\forall t \leq \bar{t}$, $k^*(t) \leq k^{fb}(t)$ and $x^*(t) \leq x^{fb}(t)$; and $\forall t > \bar{t}$, $k^*(t) > k^{fb}(t)$ and $x^*(t) > x^{fb}(t)$.

(ii) The optimal entrepreneurial effort is lower than the first-best, i.e., $z^* < z^{fb}$.

Proof: See the Appendix.

To provide intuition, notice that for $t \geq t^*$ we can re-write the second-best managerial effort and capital allocation as follows:

$$x^*(t) = x^{fb}(t) + \frac{\delta\sigma\gamma}{\theta(\gamma - \theta^2)}\Big[\lambda_n^* - \mu(t/\sigma - z_n^*)\Big], \tag{6}$$

$$k^*(t) = k^{fb}(t) + \frac{\delta\sigma\gamma}{\gamma - \theta^2}\Big[\lambda_n^* - \mu(t/\sigma - z_n^*)\Big]. \tag{7}$$

The quantity $\sigma\lambda_n^*$ represents the shadow value of entrepreneurial effort which increases in σ (the dispersion of project qualities), δ (the importance of project quality) and θ (the importance of managerial effort), and decreases in γ (effort-aversion). Moreover, the optimal entrepreneurial effort is equal to σz_n^*. First, consider the case in which entrepreneurial effort has no value ($\lambda_n^* = z_n^* = 0$).[11] Since $\mu > 0$ there is underinvestment in both capital and managerial effort. The reason is that the optimal mechanism must

[11] This case is studied in Bernardo, Cai, and Luo (2001).

only strike a balance between providing incentives for managerial effort and encouraging truthful reporting. In order to induce the manager to tell the truth the firm must offer compensation (i.e., information rents) that is non-decreasing in t. Thus, the total cost to the firm of motivating managerial effort from a manager of type t includes the indirect cost of providing higher information rents to *all* managers with higher t. This makes the cost of providing incentives very high, especially for low-quality projects for which there are many projects with higher t. This distortion is captured by the inverse hazard rate function, μ, and it causes the firm to optimally provide too few incentives for managerial effort and provide too little capital (due to complementarities) relative to first-best. Moreover, the distortion from first-best is decreasing in the manager's type (since $\mu' < 0$) and vanishes for the highest type.

If the investment opportunity set is not fixed then the firm must also motivate the manager to exert entrepreneurial effort. The manager must decide how much entrepreneurial effort to exert at date 1 (prior to observing a specific t); therefore, the firm can only motivate the manager to provide such effort by offering expected compensation (expectation over t) which increases in z. This can be achieved by increasing incentive pay and capital over some range of project qualities; however, since the distortion from offering incentive pay and capital is smaller for the higher quality projects, it is relatively inexpensive for the firm to provide incentives and capital for high quality projects. Thus, for projects of sufficiently high quality, managerial effort and capital allocation is too high relative to first-best.

Corollary 1 also states that there is an under-provision of entrepreneurial effort in the optimal mechanism relative to the first-best. In the first-best,the firm chooses entrepreneurial effort to maximize the total surplus minus the effort costs. In the second-best mechanism, the firm does not directly observe the manager's effort and must use costly incentive mechanisms to get the manager to exert entrepreneurial effort; thus the cost to the firm of motivating entrepreneurial effort is higher and the optimal entrepreneurial effort is lower than the first-best level.

Corollary 2. [Comparative Statics]

(i) The optimal entrepreneurial effort z^* *increases in* σ, δ, *and* θ *and decreases in* γ.

(ii) The optimal capital allocation $k^*(t)$ *and managerial effort* $x^*(t)$ *increase* θ *and* δ, *and decrease in* γ *for sufficiently high project qualities; and increase in* σ *for high project qualities and decrease in* σ *for low project qualities, i.e., there exists* $t_1 > t^*$ *such that* $\forall t \leq t_1$, $\partial k^*/\partial\sigma \leq 0$ *and* $\partial x^*/\partial\sigma \leq 0$*; and* $\forall t > t_1$, $\partial k^*/\partial\sigma > 0$ *and* $\partial x^*/\partial\sigma > 0$.

Proof: See the Appendix.

The comparative statics for the entrepreneurial effort z^* in the optimal mechanism are identical to the first-best. Moreover, the comparative statics for the optimal managerial effort and capital allocation are identical to the first-best except with respect to the parameter σ. The reason for this is evident by inspection of optimal managerial effort and capital allocation given in equations (6) and (7). In the first-best solution, the optimal managerial effort and capital allocation are independent of σ because both the firm and the manager are risk neutral and there are no information or incentive problems. In the second-best mechanism, higher σ has two effects on the optimal managerial effort and capital allocation. First, higher σ implies a higher shadow value of entrepreneurial effort ($\sigma\lambda_n^*$) which makes it optimal to increase managerial incentives and the capital allocation. Second, higher σ exacerbates the already high cost of increasing managerial incentives and capital for low t because the *expected* cost of increasing information rents to all higher t is increasing when there is more probability mass in the tails. Thus, when σ increases it is efficient to increase (decrease) the capital allocation and provide more (less) managerial incentives for high (low) quality projects.

3.3 Incentive pay

The following proposition demonstrates that the optimal mechanism can be implemented with a linear compensation contract.

Proposition 3. *The optimal mechanism can be implemented with the capital allocation $k^*(t)$ and a linear compensation contract, $w^*(t,V) = a^*(t) + b^*(t)V$. The fixed salary is given by:*

$$a^*(t) = \bar{U} + \int_{-\infty}^{t} \delta b^* k^* ds - 0.5\delta b^* k^* t.$$

If $t < t^* = \sigma t_n^*$ then $b^*(t) = 0$ and if $t \geq t^*$ then

$$b^*(t) = \frac{\gamma x^*}{\theta k^*} = \frac{t + \frac{\sigma\gamma}{\theta^2}(\lambda_n^* - \mu(t/\sigma - z_n^*))}{t + \sigma(\lambda_n^* - \mu(t/\sigma - z_n^*))} = 1 - \frac{\frac{(\gamma-\theta^2)}{\theta^2}(\mu(t/\sigma - z_n^*) - \lambda_n^*)}{t/\sigma + \lambda_n^* - \mu(t/\sigma - z_n^*)}.$$

Proof: See the Appendix.

The intuition for our overinvestment result is clear if we consider this implementation with a linear compensation contract. We show in the Appendix that in order to induce a manager of type t to tell the truth they must be offered expected utility $U(t) \equiv U(t,t,x(t,t))$ given by:

$$U(t) = \bar{U} + \delta \int_{-\infty}^{t} b(s)k(s)ds.$$

The term $\delta \int_{-\infty}^{t} b(s)k(s)ds$ represents the information rents given to a type-t manager which are non-decreasing in t. As we argued earlier, the greater is profit-sharing and capital allocated to lower-type managers the greater are the information rents that must be given to *all* higher-type managers which makes the cost of providing profit-sharing and capital excessive, especially for low-quality projects. To motivate entrepreneurial effort, the firm must also offer expected information rents (expectation over t) which increase in z. This can be achieved by increasing the capital allocation and profit-sharing (at least in some range of t); however, since the indirect cost of giving information rents to the manager are larger for low-quality projects the firm optimally overinvests (underinvests) in capital and managerial effort for high-quality projects (low-quality) projects.

Corollary 3. *(i) For low project qualities ($t^* < t \leq t_1$), the optimal performance pay $b^*(t)$ is increasing in project quality t and decreasing in the riskiness of project σ.*

(ii) For high project qualities ($t > t_1$), the optimal performance pay $b^*(t)$ is decreasing in project quality t and increasing in the riskiness of project σ.

Proof: See the Appendix.

The intuition for the non-monotonic relation between profit-sharing and σ is as follows. As we argued earlier, higher σ increases the value of entrepreneurial effort thus the firm must increase information rents (at least in some region of t) to motivate such effort; however, higher σ increases the already high cost of offering profit-sharing for low t because the *expected* cost of increasing information rents to all higher t is increasing when there is more probability mass in the tails. Thus, it is efficient to increase (decrease) the profit-sharing for high (low) quality projects when σ increases. Interestingly, there is a robust region in which managerial profit-sharing increases with uncertainty. This prediction is contrary to the standard principal-agent model in which uncertainty and incentives are negatively related. In the standard model, a risk-neutral firm must provide incentives to motivate a risk-averse manager to provide valuable, but privately costly, effort and the only uncertainty is measurement error. The manager's effort is assumed to be additive and non-stochastic thus the marginal benefit of providing effort incentives is independent of uncertainty; however, the marginal cost of providing effort incentives is increasing in uncertainty because the manager is risk-averse. Thus, the standard model predicts that uncertainty and incentives are negatively related. Recent empirical evidence finds the opposite relation (e.g., Prendergast, 2002). In our model, however, both the benefits and costs of providing incentives depend on the uncertainty of the economic environment; in particular, when the underlying uncertainty of the business is high (measured by the dispersion in potential project qualities), motivating entrepreneurial activity is more valuable thus we get a robust region in which uncertainty and incentives are positively related.

3.4 Comparison to the case without entrepreneurial effort

In our optimal mechanism, and under assumptions (A1) and (A2), the firm must provide the manager with sufficient information rents to motivate entrepreneurial effort. This may not always be true when, for example, entrepreneurial effort is not sufficiently valuable (e.g., assumption (A2) is violated) or the firm cannot commit at date 0 to a mechanism that gives extra information rents at date 1 in order to motivate the manager to expend entrepreneurial effort. In such cases, the manager does not exert entrepreneurial effort in equilibrium. If all other aspects of our model are unaffected, then by simply setting $\lambda^* = z^* = 0$ in Propositions 2 and 3, we get the firm's optimal mechanism (in the positive range):

$$x^{**}(t) = \frac{\theta\delta}{\gamma - \theta^2}\Big[t - \frac{\sigma\gamma}{\theta^2}\mu(t/\sigma)\Big], \tag{8}$$

$$k^{**}(t) = \frac{\gamma\delta}{\gamma - \theta^2}\Big[t - \sigma\mu(t/\sigma)\Big], \tag{9}$$

$$b^{**}(t) = \frac{\gamma x^*}{\theta k^*} = \frac{t - \frac{\sigma\gamma}{\theta^2}\mu(t/\sigma)}{t - \sigma\mu(t/\sigma)} = 1 - \frac{\frac{(\gamma-\theta^2)}{\theta^2}\mu(t/\sigma)}{t/\sigma - \mu(t/\sigma)}. \tag{10}$$

Clearly, there is underinvestment in the optimal capital allocation and managerial effort relative to first-best when entrepreneurial effort is not important. Since for $t \geq \bar{t}$ we have $x^*(t) > x^{fb}(t)$ and $k^*(t) > k^{fb}(t)$, both the capital allocation and managerial effort are greater when entrepreneurial effort is valuable than when it is not for sufficiently large t. A comparison of the profit-sharing rules show that $b^*(t) > b^{**}(t)$ whenever $\lambda_n^* - \mu(t/\sigma - z_n^*) > -\mu(t/\sigma)$ which is true for sufficiently large t (since $\lim_{y\to\infty}\mu(y) = 0$). Finally, in the case of no entrepreneurial effort, $x^{**}(t)$, $k^{**}(t)$, and $b^{**}(t)$ are all decreasing in σ, which is in sharp contrast to the case when entrepreneurial effort is important.

4 Robustness

Distribution of project quality

Although we assume that project types are normally distributed in our model, the existence of an overinvestment region holds for more general distributions. For example, suppose that conditional on the entrepreneurial effort z, the project quality t has a distribution function of $F(t|z)$ and density function $f(t|z) > 0$ with a decreasing inverse hazard rate. Also, suppose that z shifts the distribution function $F(t|z)$ in the sense of first-order stochastic dominance, i.e., $F_z(t|z) \leq 0 \ \forall t$. It can be shown that Eqs. (6) and (7) become

$$x^*(t) = x^{fb}(t) + \frac{\delta\gamma}{\theta(\gamma - \theta^2)}\Big[\lambda^* - \mu(t|z)\Big], \quad (11)$$

$$k^*(t) = k^{fb}(t) + \frac{\delta\gamma}{\gamma - \theta^2}\Big[\lambda^* - \mu(t|z)\Big]. \quad (12)$$

where λ^* is the shadow value of entrepreneurial effort and $\mu(t|z) = [1 - F(t|z)]/f(t|z)$ is decreasing in t. A sufficient condition for the results in Corollary 1 to hold is that as $t \to \infty$, $\mu(t|z) \to 0$. Thus, in this more general case, there is underinvestment in capital and managerial effort for low-quality projects but overinvestment for high-quality projects.

to be completed

5 Conclusions

to be completed

Appendix

Proof of Proposition 1. The f.o.c.s for the first best solution are

$$
\begin{aligned}
0 &= \delta t + \theta x - k, \\
0 &= \theta k - \gamma x, \\
0 &= -\frac{1}{\sigma}\int_{-\infty}^{\infty}\Big[(\delta t + \theta x)k - 0.5k^2 - 0.5\gamma x^2\Big] d\phi(\frac{t-z}{\sigma}) - \gamma z.
\end{aligned}
$$

The first two f.o.c.s (for k and x) imply

$$
\begin{aligned}
x^{fb}(t) &= \frac{\theta\delta}{\gamma - \theta^2}\max(0,t), \\
k^{fb}(t) &= \frac{\gamma\delta}{\gamma - \theta^2}\max(0,t).
\end{aligned}
$$

Substituting the optimal $x^{fb}(t)$ and $k^{fb}(t)$ into the third f.o.c. (for z) yields

$$
\begin{aligned}
0 &= -0.5\frac{1}{\sigma}\frac{\gamma\delta^2}{\gamma - \theta^2}\int_{0}^{\infty} t^2 d\phi(\frac{t-z}{\sigma}) - \gamma z \\
&= \frac{\gamma\delta^2}{\gamma - \theta^2}\Big[z\Phi(\frac{z}{\sigma}) + \sigma\phi(\frac{z}{\sigma})\Big] - \gamma z.
\end{aligned}
$$

Therefore,

$$
0 = \frac{\delta^2}{\gamma}\Big[\phi(\frac{z^{fb}}{\sigma}) + \frac{z^{fb}}{\sigma}\Phi(\frac{z^{fb}}{\sigma})\Big] - (1 - \frac{\theta^2}{\gamma})\frac{z^{fb}}{\sigma}.
$$

We show that this equation has a unique and positive solution. Note that a solution to the above equation, z^{fb}/σ, does not depend on σ. Let us define a function

$$
G(y) \equiv \frac{\delta^2}{\gamma}\Big[\phi(y) + y\Phi(y)\Big] - (1 - \frac{\theta^2}{\gamma})y.
$$

It suffices to show $H(y) = 0$ has a unique and positive solution.

Note that

$$
\begin{aligned}
G' &= \frac{\delta^2}{\gamma}\Phi(y) - (1 - \frac{\theta^2}{\gamma}) < \frac{\delta^2 + \theta^2 - \gamma}{\gamma} < 0, \\
G(0) &> 0, \\
G(\infty) &= \frac{\delta^2}{\gamma}\lim_{y\to\infty}\phi(y) - (1 - \frac{\theta^2}{\gamma})\lim_{y\to\infty} y\Big[1 - \frac{\delta^2}{\gamma - \theta^2}\Phi(y)\Big] \to -\infty,
\end{aligned}
$$

where the inequalities follow from Assumption (A1). Therefore, there is a unique solution to $G(y) = 0$ and it is positive.

The properties of $k^{fb}(t)$ and $x^{fb}(t)$ are obvious. Since z^{fb}/σ does not depend on σ, z^{fb} increases in σ. Under Assumption (A1),

$$
\frac{\partial z^{fb}}{\partial(\delta^2)} = -\frac{\sigma\phi(\frac{z^{fb}}{\sigma}) + z^{fb}\Phi(\frac{z^{fb}}{\sigma})}{\delta^2\Phi(\frac{z^{fb}}{\sigma}) + \theta^2 - \gamma} > 0.
$$

Therefore, z^{fb} increases in δ. Similarly, z^{fb} increases in θ^2, and decreases in γ. Q.E.D.

Proof of Proposition 2: Our proof follows the two-step approach of Laffont and Tirole (1986). In step 1, we consider a program (denoted R) that relaxes some constraints in Program P. With less stringent constraints, headquarters should get at least the same expected payoff in Program R as in Program P. In step 2, we consider a program (denoted L) that uses compensation contracts linear in cash flows. Since Program L restricts to a narrower class of mechanisms, headquarters can get at most the same expected payoff in Program L as in Program P. Finally, we demonstrate that headquarters' expected payoffs from Programs R and L are the same. Consequently, the optimal solutions for all three Programs L, R and P must be the same.

Step 1: Consider a relaxed program, denoted R, in which headquarters can verify $\delta t + \theta\hat{x}$. In this hypothetical case, having reported $\hat{t}$, the manager must choose $\hat{x}$ so that $\delta t + \theta\hat{x} =$

$\delta\hat{t} + \theta x(\hat{t})$. Otherwise, headquarters would be sure that the manager had either lied about his true type or didn't exert the required managerial effort, and could punish him (arbitrarily) severely. Consequently, if the manager of a true type t reports $\hat{t}$, he must choose $\hat{x} = x(\hat{t}) + \frac{\delta(\hat{t}-t)}{\theta}$, resulting in the cash flows of $V(\hat{t}, \epsilon) = (\delta\hat{t} + \theta x(\hat{t}))k(\hat{t}) - 0.5k(\hat{t})^2 + \epsilon$. Compared with Program P, the (IC1) constraint is completely relaxed in Program R.

In Program R, if the manager of a true type t reports $\hat{t}$, his expected payoff is

$$U(t, \hat{t}) \quad = \quad E_\epsilon w(\hat{t}, V(\hat{t}, \epsilon)) - 0.5\gamma(x(\hat{t}) + \frac{\delta(\hat{t} - t)}{\theta})^2.$$

By the Envelope Theorem,

$$\frac{dU(t,t)}{dt} \quad = \quad \frac{\partial U(t,\hat{t})}{\partial t}|_{\hat{t}=t} + \frac{\partial U(t,\hat{t})}{\partial \hat{t}}|_{\hat{t}=t} = \frac{\partial U(t,\hat{t})}{\partial t}|_{\hat{t}=t} = \frac{\delta\gamma x(t)}{\theta}.$$

Integrating yields

$$U(t) = U(-\infty) + \int_{-\infty}^{t} \frac{\delta\gamma x(s)}{\theta} ds.$$

This is the manager's expected payoff after he has expanded entrepreneurial effort z and has observed the realization of project quality t at date 1. The effort cost of $0.5\gamma z^2$ is sunk. Imposing $U(-\infty) = \bar{U}$ from the (IR) constraint, and taking expectation yields

$$EU = \bar{U} + \int_{-\infty}^{\infty} \frac{\delta\gamma x(t)}{\theta}(1 - \Phi(\frac{t - \hat{z}}{\sigma}))dt.$$

The (IC3) constraint requires

$$z \in \arg\max_{\hat{z}} \bar{U} + \int_{-\infty}^{\infty} \frac{\delta\gamma x(t)}{\theta}(1 - \Phi(\frac{t - \hat{z}}{\sigma}))dt - 0.5\gamma\hat{z}^2.$$

The first order condition for z is

$$\int_{-\infty}^{\infty} \frac{\delta\gamma x(t)}{\theta}\frac{1}{\sigma}\phi(\frac{t-z}{\sigma})dt - \gamma z = 0. \tag{13}$$

Substituting the expression for wage, $Ew = EU + 0.5\gamma E(x^2)$, into the headquarters' expected payoff yields

$$\begin{aligned} E\Pi &= \int_{-\infty}^{\infty} (E_\epsilon V - 0.5\gamma x^2 - U)d\Phi(\frac{t-z}{\sigma}) \\ &= \int_{-\infty}^{\infty} \Big[(\delta tk + \theta xk - 0.5k^2 - 0.5\gamma x^2)\frac{1}{\sigma}\phi(\frac{t-z}{\sigma}) - \frac{\delta\gamma x}{\theta}(1-\Phi(\frac{t-z}{\sigma}))\Big]dt - \bar{U}. \end{aligned}$$

Denote λ the Lagrangian multiplier for the IC constraint for z (Eq. 13). The headquarters' maximum expected payoff can be expressed as:

$$\begin{aligned} E\Pi^R &= \max_{z,k(t),x(t)\geq 0,\lambda} \int_{-\infty}^{\infty} \Big[(\delta tk + \theta xk - 0.5k^2 - 0.5\gamma x^2)\frac{1}{\sigma}\phi(\frac{t-z}{\sigma}) - \frac{\delta\gamma x}{\theta}(1-\Phi(\frac{t-z}{\sigma}))\Big]dt \\ &\quad -\lambda(\gamma z - \int_{-\infty}^{\infty} \frac{\delta\gamma x}{\theta}\frac{1}{\sigma}\phi(\frac{t-z}{\sigma})dt) - \bar{U}. \end{aligned} \tag{14}$$

Since Program R provides more flexibility to headquarters to discipline the manager, it should yield headquarters at least the same expected payoff as Program P does, i.e., $E\Pi^R \geq E\Pi^P$.

Step 2: Consider a program, denoted L, in which the headquarters is constrained to use compensation contracts linear in cash flows: $w(\hat{t}, V) = a(\hat{t}) + b(\hat{t})V$. Under this contract, the manager of type t can get the following expected payoff from announcing $\hat{t}$ and choosing managerial effort $\hat{x}$:

$$U(t,\hat{t},\hat{x}) = a(\hat{t}) + b(\hat{t})\Big[(\delta t + \theta\hat{x})k(\hat{t}) - 0.5k(\hat{t})^2\Big] - 0.5\gamma\hat{x}.$$

The first order condition for $\hat{x}$ is

$$\theta b(\hat{t})k(\hat{t}) - \gamma\hat{x} = 0.$$

Thus $\hat{x} = \theta b(\hat{t})k(\hat{t})/\gamma$. The second order condition holds obviously.

Note that the $\hat{x}$ depends only on $\hat{t}$. By the Envelope Theorem,

$$\frac{dU(t,t)}{dt} = \delta b(t)k(t).$$

Thus

$$U(t) = U(-\infty) + \int_{-\infty}^{t} \delta b(s)k(s)ds. \tag{15}$$

Imposing $U(-\infty) = \bar{U}$ and taking expectation yields

$$\begin{aligned} EU &= \bar{U} + \int_{-\infty}^{\infty} \delta b(t)k(t)(1 - \Phi(\frac{t-\hat{z}}{\sigma}))dt \\ &= \bar{U} + \int_{-\infty}^{\infty} \frac{\delta\gamma x(t)}{\theta}(1 - \Phi(\frac{t-\hat{z}}{\sigma}))dt, \end{aligned}$$

where the last equality obtains by substituting the expression for $x(t)$.

The (IC3) constraint requires

$$z \in \arg\max_{\hat{z}} \bar{U} + \int_{-\infty}^{\infty} \frac{\delta\gamma x(t)}{\theta}(1 - \Phi(\frac{t-\hat{z}}{\sigma}))dt - 0.5\gamma\hat{z}^2.$$

The first order condition for z is

$$\int_{-\infty}^{\infty} \frac{\delta\gamma x(t)}{\theta}\frac{1}{\sigma}\phi(\frac{t-z}{\sigma})dt - \gamma z = 0. \tag{16}$$

The s.o.c. for $\hat{z}$ can be easily satisfied holds as long as δ is sufficiently small.

Substituting the expression for wages, $Ew = EU + 0.5\gamma E(x^2)$, into the headquarters' expected payoff yields

$$
\begin{aligned}
E\Pi &= \int_{-\infty}^{\infty} (V - 0.5\gamma x^2 - U) d\Phi(\frac{t-z}{\sigma}) \\
&= \int_{-\infty}^{\infty} \Big[(\delta tk + \theta xk - 0.5k^2 - 0.5\gamma x^2)\frac{1}{\sigma}\phi(\frac{t-z}{\sigma}) - \frac{\delta\gamma x}{\theta}(1 - \Phi(\frac{t-z}{\sigma}))\Big] dt - \bar{U}.
\end{aligned}
$$

Denote λ as the Lagrangian multiplier for the IC constraint for z (Eq. 16). The headquarters' mechanism design problem can be expressed as:

$$
\begin{aligned}
E\Pi^L &= \max_{z,k(t),x(t)\geq 0,\lambda} \int_{-\infty}^{\infty} \Big[(\delta tk + \theta xk - 0.5k^2 - 0.5\gamma x^2)\frac{1}{\sigma}\phi(\frac{t-z}{\sigma}) - \frac{\delta\gamma x}{\theta}(1 - \Phi(\frac{t-z}{\sigma}))\Big] dt \\
&\qquad -\lambda\Big[\gamma z - \int_{-\infty}^{\infty} \frac{\delta\gamma x}{\theta}\frac{1}{\sigma}\phi(\frac{t-z}{\sigma}) dt\Big] - \bar{U}. \qquad (17)
\end{aligned}
$$

Since the Program L allows only a subset of mechanisms of the Program P, it must be that $E\Pi^L \leq E\Pi^P$. However, comparing Eqs. (14) and (17) shows that $E\Pi^L = E\Pi^R \geq E\Pi^P$. Therefore, $E\Pi^L = E\Pi^P = E\Pi^R$, so the optimal mechanism with linear compensation contracts cannot be further improved upon.

Optimal Mechanism: We now solve the optimal mechanism. To simplify notation, we scale down t by σ so that the new state variable ($t_n = t/\sigma$) has a normal distribution with mean $z_n = z/\sigma$ and variance 1. Using these variable transformations to rewrite the objective function (Eq. 17), we have

$$
\begin{aligned}
E\Pi &= \max_{z,k(t),x(t)} \int_{-\infty}^{\infty} \Big[(\delta tk + \theta xk - 0.5k^2 - 0.5\gamma x^2)\phi(t_n - z_n) - \frac{\sigma\delta\gamma x}{\theta}(1 - \Phi(t_n - z_n))\Big] dt_n \\
&\qquad -\lambda\Big[\gamma z - \int_{-\infty}^{\infty} \frac{\delta\gamma x}{\theta}\phi(t_n - z_n) dt_n\Big] - \bar{U}.
\end{aligned}
$$

The first order conditions for interior solutions are

$$
\begin{aligned}
0 &= \delta t + \theta x - k, \\
0 &= \theta k - \gamma x - \frac{\sigma\delta\gamma}{\theta}\mu(t_n - z_n) + \lambda\frac{\delta\gamma}{\theta}, \\
0 &= \int_{-\infty}^{\infty}\Big[(\delta t k + \theta x k - 0.5k^2 - 0.5\gamma x^2)(t_n - z_n) - \frac{\sigma\delta\gamma x}{\theta}\Big]\phi(t_n - z_n)dt_n \\
&\quad -\lambda\Big[\sigma\gamma - \int_{-\infty}^{\infty}\frac{\delta\gamma x}{\theta}(t_n - z_n)\phi(t_n - z_n)dt_n\Big], \\
0 &= \int_{-\infty}^{\infty}\frac{\delta\gamma x}{\theta}\phi(t_n - z_n)dt_n - \gamma z.
\end{aligned}
$$

where the third equation uses the fact that $d\phi(t_n - z_n)/dz_n = (t_n - z_n)\phi(t_n - z_n)$.

Define t_n^* such that $t_n^* + \frac{\gamma}{\theta^2}(\lambda/\sigma - \mu(t_n^* - z_n)) = 0$. It can be shown that $t_n^* \geq 0$ (Lemma 1 below). For $t_n \geq t_n^*$, the first two first order conditions (with respect to k and x) imply that

$$
\begin{aligned}
x^*(t_n) &= \frac{\sigma\theta\delta}{\gamma - \theta^2}\Big[t_n + \frac{\gamma}{\theta^2}(\lambda/\sigma - \mu(t_n - z_n))\Big], \\
k^*(t_n) &= \frac{\sigma\gamma\delta}{\gamma - \theta^2}\Big[t_n + \lambda/\sigma - \mu(t_n - z_n)\Big];
\end{aligned}
$$

for $t_n < t_n^*$, then $x^* = 0$ and $k^* = \max(0, \delta t)$.

Substituting x^* and k^* into the fourth first order condition (with respect to λ) yields

$$
\begin{aligned}
0 &= \int_{t_n^*}^{\infty}\frac{\delta x^*}{\theta}\phi(t_n - z_n)dt_n - z \\
&= \frac{\sigma\delta^2}{\gamma - \theta^2}\int_{t_n^*}^{\infty}\Big[t_n + \frac{\gamma}{\theta^2}(\lambda/\sigma - \mu(t_n - z_n))\Big]\phi(t_n - z_n)dt_n - \sigma z_n
\end{aligned}
$$

Using the facts that (i) $1 - \Phi(y) = \Phi(-y)$; and (ii) $\int y\phi(y)dy = -\int d\phi(y) = -\phi(y)$; and (iii) $\int(1 - \Phi(y))dy = y(1 - \Phi(y)) + \int y\phi(y)dy = y(1 - \Phi(y)) - \phi(y)$, we can rewrite the above equation as

$$
\begin{aligned}
z_n &= \frac{\delta^2}{\gamma-\theta^2}\Big[\phi(t_n^*-z_n)+(z_n+\frac{\lambda\gamma}{\sigma\theta^2})\Phi(z_n-t_n^*)+\frac{\gamma}{\theta^2}(t_n^*-z_n)(1-\Phi(t_n^*-z_n))-\frac{\gamma}{\theta^2}\phi(t_n^*-z_n)\Big] \\
&= \frac{\delta^2}{\theta^2}\left[\frac{\gamma}{\gamma-\theta^2}(t_n^*+\frac{\lambda}{\sigma})\Phi(z_n-t_n^*)-\Big[\phi(z_n-t_n^*)+z_n\Phi(z_n-t_n^*)\Big]\right]. \qquad (18)
\end{aligned}
$$

Substituting the optimal x^* and k^* into the third f.o.c (with respect to z_n) yields

$$
\begin{aligned}
0 &= -\int_0^\infty\Big[0.5(k^{*2}-\gamma x^{*2})d\phi(t_n-z_n)+\frac{\delta\sigma\gamma x^*}{\theta}\phi(t_n-z_n)dt_n\Big] \\
&\quad -\lambda\Big[\gamma\sigma+\frac{\delta\gamma}{\theta}\int_{t_n^*}^\infty x^*d\phi(t_n-z_n)\Big] \\
&= \int_0^\infty\Big[0.5\phi(t_n-z_n)d(k^{*2}-\gamma x^{*2})-\frac{\delta\sigma\gamma x^*}{\theta}\phi(t_n-z_n)dt_n\Big] \\
&\quad -\lambda\Big[\gamma\sigma+\frac{\delta\gamma}{\theta}\int_{t_n^*}^\infty x^*d\phi(t_n-z_n)\Big] \\
&= \int_0^\infty(k^*k^{*\prime}-\gamma x^*x^{*\prime}-\frac{\delta\sigma\gamma x^*}{\theta})\phi(t_n-z_n)dt_n-\lambda\Big[\gamma\sigma+\frac{\delta\gamma}{\theta}\int_{t_n^*}^\infty x^*d\phi(t_n-z_n)\Big] \\
&= \int_{t_n^*}^\infty(k^*k^{*\prime}-\gamma x^*x^{*\prime}-\frac{\delta\sigma\gamma x^*}{\theta})\phi(t_n-z_n)dt_n \\
&\quad +\delta^2\sigma^2\int_0^{t_n^*}t_n\phi(t_n-z_n)dt_n-\lambda\Big[\gamma\sigma+\frac{\delta\gamma}{\theta}\int_{t_n^*}^\infty x^*d\phi(t_n-z_n)\Big]
\end{aligned}
$$

Using the shortcut notation $\mu=\mu(t_n-z_n)$, we have, for $t\geq t_n^*$,

$$
\begin{aligned}
&k^*k^{*\prime}-\gamma x^*x^{*\prime}-\frac{\delta\sigma\gamma x^*}{\theta} \\
&= \frac{\sigma^2\gamma^2\delta^2}{(\gamma-\theta^2)^2}\Big[t_n+\lambda/\sigma-\mu\Big](1-\mu')-\frac{\gamma\sigma^2\theta^2\delta^2}{(\gamma-\theta^2)^2}\Big[t_n+\frac{\gamma}{\theta^2}(\lambda/\sigma-\mu)\Big](1-\frac{\gamma}{\theta^2}\mu') \\
&\quad -\frac{\sigma^2\gamma\delta^2}{\gamma-\theta^2}\Big[t_n+\frac{\gamma}{\theta^2}(\lambda/\sigma-\mu)\Big] \\
&= \frac{\sigma^2\gamma\delta^2}{\gamma-\theta^2}t_n-\mu'\frac{\gamma^2\sigma^2\delta^2}{(\gamma-\theta^2)^2}(1-\frac{\gamma}{\theta^2})(\lambda/\sigma-\mu)-\frac{\sigma^2\gamma\delta^2}{\gamma-\theta^2}\Big[t_n+\frac{\gamma}{\theta^2}(\lambda/\sigma-\mu)\Big] \\
&= \frac{\gamma^2\sigma^2\delta^2}{\theta^2(\gamma-\theta^2)}(\lambda/\sigma-\mu)(\mu'-1)
\end{aligned}
$$

Plugging this back into the first order condition with respect to z_n gives

$$
\begin{aligned}
&= \frac{\gamma^2\sigma^2\delta^2}{\theta^2(\gamma-\theta^2)}\int_{t_n^*}^{\infty}(\lambda/\sigma-\mu)(\mu'-1)\phi(t_n-z_n)dt_n - \frac{\lambda\delta\gamma}{\theta}\int_{t_n^*}^{\infty}x^*d\phi(t_n-z_n) \\
&\quad +\delta^2\sigma^2\Big[\int_0^{t_n^*}z_nd\Phi(t_n-z_n)-\int_0^{t_n^*}d\phi(t_n-z_n)\Big]-\lambda\gamma\sigma \\
&= \frac{\gamma^2\sigma^2\delta^2}{\theta^2(\gamma-\theta^2)}\int_{t_n^*}^{\infty}(\lambda/\sigma-\mu)(\mu'-1)\phi(t_n-z_n)dt_n + \frac{\lambda\delta\gamma}{\theta}\int_{t_n^*}^{\infty}x^{*\prime}\phi(t_n-z_n)dt_n \\
&\quad +\delta^2\sigma^2\Big[z_n(\Phi(t_n^*-z_n)-\Phi(-z_n))-\phi(t_n^*-z_n)+\phi(-z_n)\Big]-\lambda\gamma\sigma \\
&= \frac{\gamma^2\sigma^2\delta^2}{\theta^2(\gamma-\theta^2)}\int_{t_n^*}^{\infty}\Big[-\frac{\lambda}{\sigma}\frac{\gamma-\theta^2}{\gamma}-\mu\mu'+\mu\Big]\phi(t_n-z_n)dt_n \\
&\quad +\delta^2\sigma^2\Big[z_n(\Phi(t_n^*-z_n)-\Phi(-z_n))-\phi(t_n^*-z_n)+\phi(-z_n)\Big]-\lambda\gamma\sigma \\
&= \frac{\gamma^2\sigma^2\delta^2}{\theta^2(\gamma-\theta^2)}\int_{t_n^*}^{\infty}\Big[-\frac{\lambda}{\sigma}\frac{\gamma-\theta^2}{\gamma}-\mu\mu'+\mu\Big]\phi(t_n-z_n)dt_n \\
&\quad +\delta^2\sigma^2\Big[z_n(\Phi(t_n^*-z_n)-\Phi(-z_n))-\phi(t_n^*-z_n)+\phi(-z_n)\Big]-\lambda\gamma\sigma \\
&= \frac{\lambda\gamma\sigma\delta^2}{\theta^2}\Phi(z_n-t_n^*)-\frac{\gamma^2\sigma^2\delta^2}{\theta^2(\gamma-\theta^2)}\Big[\int_{t_n^*}^{\infty}(1-\Phi(t_n-z_n))d\mu-\int_{t_n^*}^{\infty}\mu\phi(t_n-z_n)dt_n\Big] \\
&\quad +\delta^2\sigma^2\Big[z_n(\Phi(z_n)-\Phi(z_n-t_n^*))-\phi(z_n-t_n^*)+\phi(z_n)\Big]-\lambda\gamma\sigma \\
&= -\frac{\lambda\gamma\sigma\delta^2}{\theta^2}\Phi(z_n-t_n^*)+\frac{\gamma^2\sigma^2\delta^2}{\theta^2(\gamma-\theta^2)}(1-\Phi(t_n^*-z_n))\mu(t_n^*-z_n) \\
&\quad +\delta^2\sigma^2\Big[z_n(\Phi(z_n)-\Phi(z_n-t_n^*))-\phi(z_n-t_n^*)+\phi(z_n)\Big]-\lambda\gamma\sigma
\end{aligned}
$$

By the definition of t_n^*, $\mu(t_n^*-z_n)=\theta^2t_n^*/\gamma+\lambda/\sigma$. So we continue

$$
\begin{aligned}
&= -\frac{\lambda\gamma\sigma\delta^2}{\theta^2}\Phi(z_n-t_n^*)+\frac{\gamma\sigma^2\delta^2}{\gamma-\theta^2}\Phi(z_n-t_n^*)t_n^*+\frac{\lambda\gamma^2\sigma\delta^2}{\theta^2(\gamma-\theta^2)}\Phi(z_n-t_n^*) \\
&\quad +\delta^2\sigma^2\left[-\Big[\phi(z_n-t_n^*)+z_n\Phi(z_n-t_n^*)\Big]+\Big[\phi(z_n)+z_n\Phi(z_n)\Big]\right]-\lambda\gamma\sigma \\
&= \delta^2\sigma^2\left[\frac{\gamma}{\gamma-\theta^2}(t_n^*+\frac{\lambda}{\sigma})\Phi(z_n-t_n^*)-\Big[\phi(z_n-t_n^*)+z_n\Phi(z_n-t_n^*)\Big]+\Big[\phi(z_n)+z_n\Phi(z_n)\Big]\right] \\
&\quad -\lambda\gamma\sigma
\end{aligned}
$$

From Equation 18, the above equation can be simplified as

$$\lambda = \frac{\delta^2 \sigma}{\gamma}\Big[\phi(z_n) + z_n \Phi(z_n)\Big] + \frac{\sigma\theta^2}{\gamma} z_n$$

Define $\lambda_n = \lambda/\sigma$. Then, the solution to the mechanism design problem, $(\lambda_n^*, z_n^*, t_n^*)$, satisfies the following equations:

$$\lambda_n^* = \frac{\delta^2}{\gamma}\Big[\phi(z_n^*) + z_n^* \Phi(z_n^*)\Big] + \frac{\theta^2}{\gamma} z_n^* \tag{19}$$

$$z_n^* = \frac{\delta^2}{\theta^2}\left[\frac{\gamma}{\gamma - \theta^2}(t_n^* + \lambda_n^*)\Phi(z_n^* - t_n^*) - \Big[\phi(z_n^* - t_n^*) + z_n^* \Phi(z_n^* - t_n^*)\Big]\right] \tag{20}$$

$$t_n^* = \frac{\gamma}{\theta^2}\left[\mu(t_n^* - z_n^*) - \lambda_n^*\right] \tag{21}$$

Clearly, if a solution exists, $(\lambda_n^*, z_n^*, t_n^*)$ are independent of σ. From such a solution, we have $z^* = \sigma z_n^*$, $t^* = \sigma t_n^*$ and $\lambda^* = \sigma \lambda_n^*$. Then, for $t \geq t^* = \sigma t_n^*$, we have

$$x^*(t) = \frac{\theta\delta}{\gamma - \theta^2}\Big[t + \frac{\sigma\gamma}{\theta^2}(\lambda_n^* - \mu(t/\sigma - z_n^*))\Big], \tag{22}$$

$$k^*(t) = \frac{\gamma\delta}{\gamma - \theta^2}\Big[t + \sigma(\lambda_n^* - \mu(t/\sigma - z_n^*))\Big]. \tag{23}$$

In Lemma 1 below, we show that a solution to Eqs (19)-(21) indeed exists and is unique and positive.

Finally, it is straightforward to verify that $x^*(t)$ and $k^*(t)$ are increasing in t, so the first order approach is valid. Q.E.D.

Lemma 1. *Eqs (19)-(21) have a unique solution* $(\lambda_n^*, z_n^*, t_n^*)$*, where* $\lambda_n^*, z_n^*, t_n^* > 0$*.*

Proof: From Eq. (19), λ_n^* is a strictly increasing function of z_n^*, because

$$\frac{d\lambda_n^*}{dz_n^*} = \frac{\delta^2}{\gamma}\Phi(z_n^*) + \frac{\theta^2}{\gamma} > 0$$

Furthermore, since $\lambda_n^*(0) = \frac{\delta^2}{\gamma}\phi(0) > 0$ at $z_n^* = 0$, λ_n^* must be strictly positive.

From Eq. (21), let us define

$$\begin{aligned} I &= t_\sigma^* + \frac{\gamma}{\theta^2}\lambda_\sigma^* - \frac{\gamma}{\theta^2}\mu(t_\sigma^* - z_\sigma^*) \\ &= t_\sigma^* + \frac{\gamma}{\theta^2}\left[\frac{\delta^2}{\gamma}\Big[\phi(z_\sigma^*) + z_\sigma^*\Phi(z_\sigma^*)\Big] + \frac{\theta^2}{\gamma}z_\sigma^* - \mu(t_\sigma^* - z_\sigma^*)\right] \end{aligned}$$

Since $\mu' < 0$, $\partial I/\partial t_\sigma^* > 0$. In addition, for any given z_σ^*, clearly $I(-\infty) < 0$ and $I(\infty) > 0$, since $\lim_{y\to-\infty}\mu(y) = \infty$ and $\lim_{y\to\infty}\mu(y) = 0$. Thus, for any given z_σ^*, the solution for $I(t_\sigma^*, z_\sigma^*) = 0$, t_σ^*, is uniquely determined. Consider $I_0(z_\sigma^*) = I_0(t_\sigma^* = 0, z_\sigma^*)$. Note that $I_0(0) = \frac{\gamma}{\theta^2}[\frac{\delta^2}{\gamma}\phi(0) - \mu(0)] < \frac{\gamma}{\theta^2}[\phi(0) - \mu(0)] < 0$, because $\phi(0) = 1/\sqrt{2\pi} < 0.5$ and $\mu(0) = \Phi(0)/\phi(0) = 0.5/\phi(0) > 1$. Moreover,

$$\begin{aligned} \frac{\theta^2}{\gamma}I_0' &= \frac{\delta^2}{\gamma}\Phi(z_\sigma^*) + \frac{\theta^2}{\gamma} + \mu'(-z_\sigma^*) \\ &= \frac{\delta^2}{\gamma}\Phi(z_\sigma^*) + \frac{\theta^2}{\gamma} - 1 - z_\sigma^*\mu(-z_\sigma^*) \\ &< \frac{\delta^2 + \theta^2}{\gamma} - 1 - z_\sigma^*\mu(-z_\sigma^*) \\ &< 0 \end{aligned}$$

where the second equation above uses the fact that $\mu'(y) = -1 + y\mu(y)$. It follows that for all z_σ^*, $I_0(z_\sigma^*) < 0$. Therefore, the solution for $I(t_\sigma^*, z_\sigma^*) = 0$, t_σ^*, must be positive.

From Eq. (20), we define

$$\begin{aligned} H(z_n^*) &= \frac{\delta^2}{\theta^2}\left[\frac{\gamma}{\gamma - \theta^2}(t_n^* + \lambda_n^*)\Phi(z_n^* - t_n^*) - \Big[\phi(z_n^* - t_n^*) + z_n^*\Phi(z_n^* - t_n^*)\Big]\right] - z_n^* \\ &= \frac{\delta^2}{\theta^2}\left[\frac{\theta^2}{\gamma - \theta^2}(t_n^* + \frac{\gamma}{\theta^2}\lambda_n^*)\Phi(z_n^* - t_n^*) - \Big[\phi(z_n^* - t_n^*) + (z_n^* - t_n^*)\Phi(z_n^* - t_n^*)\Big]\right] - z_n^* \end{aligned}$$

Note that

$$
\begin{aligned}
\frac{\partial H}{\partial t_n^*} &= \frac{\delta^2}{\theta^2}\left[\frac{\theta^2}{\gamma-\theta^2}\Phi(z_n^*-t_n^*) - \frac{\theta^2}{\gamma-\theta^2}(t_n^* + \frac{\gamma}{\theta^2}\lambda_\sigma^*)\phi(z_n^*-t_n^*) + \Phi(z_n^*-t_n^*)\right] \\
&= \frac{\delta^2}{\theta^2}\left[\frac{\gamma}{\gamma-\theta^2}\Phi(z_n^*-t_n^*) - \frac{\theta^2}{\gamma-\theta^2}\frac{\gamma}{\theta^2}\mu(t_n^*-z_n)\phi(z_n^*-t_n^*)\right] \\
&= \frac{\delta^2}{\theta^2}\left[\frac{\gamma}{\gamma-\theta^2}\Phi(z_n^*-t_n^*) - \frac{\gamma}{\gamma-\theta^2}\Phi(z_n^*-t_n^*)\right] \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial H}{\partial z_n^*} &= \frac{\delta^2}{\theta^2}\left[\frac{\theta^2}{\gamma-\theta^2}(t_n^* + \frac{\gamma}{\theta^2}\lambda_\sigma^*)\phi(z_n^*-t_n^*) - \Phi(z_n^*-t_n^*)\right] - 1 \\
&= \frac{\delta^2}{\theta^2}\left[\frac{\theta^2}{\gamma-\theta^2}\frac{\gamma}{\theta^2}\mu(t_n^*-z_n^*)\phi(z_n-t_n^*) - \Phi(z_n^*-t_n^*)\right] - 1 \\
&= \frac{\delta^2}{\theta^2}\left[\frac{\gamma}{\gamma-\theta^2}\Phi(z_n^*-t_n^*) - \Phi(z_n^*-t_n^*)\right] - 1 \\
&= \frac{\delta^2}{\gamma-\theta^2}\Phi(z_n^*-t_n^*) - 1
\end{aligned}
$$

$$
\frac{\partial H}{\partial \lambda_n^*} = \frac{\delta^2}{\theta^2}\frac{\gamma}{\gamma-\theta^2}\Phi(z_n^*-t_n^*)
$$

Therefore,

$$
\begin{aligned}
\frac{dH}{dz_n^*} &= \frac{\partial H}{\partial z_n^*} + \frac{\partial H}{\partial \lambda_n^*}\frac{d\lambda_n^*}{dz_n^*} + \frac{\partial H}{\partial t_n^*}\frac{dt_n^*}{dz_n^*} \\
&= \frac{\delta^2}{\theta^2}\left[\frac{\theta^2}{\gamma-\theta^2}\Phi(z_n^*-t_n^*)\right] - 1 + \frac{\delta^2}{\theta^2}\frac{\gamma}{\gamma-\theta^2}\Phi(z_n^*-t_n^*)\Big[\frac{\delta^2}{\gamma}\Phi(z_n^*) + \frac{\theta^2}{\gamma}\Big] \\
&= \frac{\delta^2}{\theta^2}\left[\frac{2\theta^2}{\gamma-\theta^2}\Phi(z_n^*-t_n^*) + \frac{\delta^2}{\gamma-\theta^2}\Phi(z_n^*-t_n^*)\Phi(z_n^*)\right] - 1 \\
&< \frac{\delta^2}{\theta^2}\frac{2\theta^2+\delta^2}{\gamma-\theta^2} - 1 \\
&< 0,
\end{aligned}
$$

where the last inequality follows from Assumption (A1).

At $z_n^* = 0$, $\lambda_n^* = \frac{\delta^2}{\gamma}\phi(0)$ and t_n^* is given by $t_n^* = \frac{\gamma}{\theta^2}(\mu(t_n^*) - \lambda_n^*)$. It can be shown that at $z_n^* = 0$, $t_n^* > 0.75$. Let $J = t_n^* - \frac{\gamma}{\theta^2}(\mu(t_n^*) - \lambda_n^*)$. Then $J(0.75) = 0.75 - \frac{\gamma}{\theta^2}(\mu(0.75) - \frac{\delta^2}{\gamma}\phi(0)) = \frac{1}{\theta^2}[0.75\theta^2 + \delta^2\phi(0) - \gamma\mu(0.75)] < 0$, since $\mu(0.75) = 0.752$. Since $J' > 0$, so the solution to $J(t_n^*) = 0$, t_n^* must be greater than 0.75.

We then have

$$
\begin{aligned}
H(0) &= \frac{\delta^2}{\theta^2}\left[\frac{\gamma}{\gamma-\theta^2}(t_n^* + \lambda^*{}_n)\Phi(-t_n^*) - \phi(t_n^*)\right] \\
&= \frac{\delta^2}{\theta^2}\phi(t_n^*)\left[\frac{\gamma}{\gamma-\theta^2}(t_n^* + \lambda_n^*)\mu(t_n^*) - 1\right] \\
&= \frac{\delta^2}{\theta^2}\phi(-t_n^*)\left[\frac{\gamma}{\gamma-\theta^2}(t_n^* + \lambda_n^*)(\frac{\theta^2}{\gamma}t_n^* + \lambda_n^*) - 1\right] \\
&= \frac{\delta^2}{\theta^2(\gamma-\theta^2)}\phi(-t_n^*)\left[\theta^2(t_n^*)^2 + (\theta^2+\gamma)\lambda_n^* t_n^* + \gamma(\lambda_n^*)^2 - \gamma + \theta^2\right] \\
&> \frac{\delta^2}{\theta^2(\gamma-\theta^2)}\phi(-t_n^*)\left[25\theta^2/16 + 0.75(\theta^2/\gamma+1)\delta^2\phi(0) + \delta^4\phi^2(0)/\gamma - \gamma\right] \\
&> 0
\end{aligned}
$$

where the last inequality follows from Assumption (A2). Therefore, $H(0) > 0$.

As $z_n^* \to \infty$, clearly $\lambda_n^* \to \infty$. From Eq. (21), it must be that $t_n^* \to \infty$. Otherwise, it can be shown that $\lim_{z_n^*\to\infty}[\mu(t_n^* - z_n^*) - \lambda_n^*] = \lim_{z_n^*\to\infty} z_n^*[\mu(t_n^* - z_n^*)/z_n^* - \lambda_n^*/z_n^*] \to \infty$, which is a contradiction with a finite t_n^* on the left hand side of Eq. (21). To show this, simply note that $\lim_{z_n^*\to\infty}\frac{\mu(t_n^*-z_n^*)}{z_n^*} = \lim_{z_n^*\to\infty}[1 + \mu(t_n^* - z_n^*)(z_n^* - t_n^*)] = \infty$, and that $\lim_{z_n^*\to\infty}\frac{\lambda_n^*}{z_n^*} = \frac{\theta^2+\delta^2}{\gamma}$. Since $t_n^* \to \infty$, it follows from Eq. (21) that $\mu(t_n^* - z_n^*) \to \infty$ and hence $z_n^* - t_n^* \to \infty$.

We have

$$
\begin{aligned}
\frac{\theta^2}{\delta^2}\lim_{z_n^*\to\infty}\frac{H}{z_n^*} &= \lim_{z_n^*\to\infty}\left[\frac{\gamma}{\gamma-\theta^2}\frac{t_n^*+\lambda_n^*}{z_n^*}\Phi(z_n^* - t_n^*) - \frac{\left[\phi(z_n^* - t_n^*) + z_n^*\Phi(z_n^* - t_n^*)\right]}{z_n^*}\right] - \frac{\theta^2}{\delta^2} \\
&= \frac{\gamma}{\gamma-\theta^2}\lim_{z_n^*\to\infty}\frac{t_n^*+\lambda_n^*}{z_n^*} - 1 - \frac{\theta^2}{\delta^2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\gamma}{\gamma-\theta^2}\lim_{z_n^*\to\infty}\frac{z_n^*+\frac{\delta^2}{\gamma}\Big[\phi(z_n^*)+z_n^*\Phi(z_n^*)\Big]+\frac{\theta^2}{\gamma}z_n^*}{z_n^*}-1-\frac{\theta^2}{\delta^2}\\
&= \frac{\gamma}{\gamma-\theta^2}\frac{\gamma+\theta^2+\delta^2}{\gamma}-1-\frac{\theta^2}{\delta^2}\\
&< 0,
\end{aligned}
$$

where the first inequality follows that $t_n^* < z_n^*$ in the limit since $z_n^* - t_n^* \to \infty$, and the last inequality holds under Assumption (A1). Therefore, $J(\infty) < 0$.

To summarize, $H' < 0$, $H(0) > 0$, $H(\infty) < 0$. Therefore, z_n^* is uniquely determined and is positive. Q.E.D.

Proof of Corollary 1. (i) Define $\bar{t}$ such that $\mu(\bar{t}/\sigma - z_n^*) - \lambda_n^* = 0$. Note that

$$
\mu(\bar{t}/\sigma - z_n^*) - \lambda_n^* = 0 < \frac{\theta^2}{\gamma}t_n^* = \mu(t_n^* - z_n^*) - \lambda_n^*
$$

From $\mu' < 0$, it must be $\bar{t} > \sigma t_n^* = t^*$. Since $k^*(\bar{t}) = k^{fb}(\bar{t})$, and

$$
k^*(\bar{t}) - k^{fb}(\bar{t}) = \frac{\sigma\gamma\delta}{\gamma-\theta^2}(\lambda_n^* - \mu(t/\sigma - z_n^*))
$$

which is increasing in t. Therefore, $k^*(t) > k^{fb}(t)$ if and only if $t > \bar{t}$. Similarly, we obtain the result for managerial efforts.

(ii) Let $z_n^{fb} = z^{fb}/\sigma$, then z_n^{fb} is the solution to the following equation

$$
z_n^{fb} = \frac{\delta^2}{\gamma}\Big[\phi(z_n^{fb}) + z_n^{fb}\Phi(z_n^{fb})\Big] + \frac{\theta^2}{\gamma}z_n^{fb}
$$

From Eqn. (19), z_n^* is the solution to the following equation

$$
\lambda_n^* = \frac{\delta^2}{\gamma}\Big[\phi(z_n^*) + z_n^*\Phi(z_n^*)\Big] + \frac{\theta^2}{\gamma}z_n^*
$$

Suppose $\lambda_n^* > z_n^*$. Then

$$
\begin{aligned}
\frac{\delta^2}{\gamma}\Big[\phi(z_n^*) + z_n^*\Phi(z_n^*)\Big] + \frac{\theta^2}{\gamma}z_n^* - z_n^* &> \frac{\delta^2}{\gamma}\Big[\phi(z_n^*) + z_n^*\Phi(z_n^*)\Big] + \frac{\theta^2}{\gamma}z_n^* - \lambda_n^* \\
&= 0 \\
&= \frac{\delta^2}{\gamma}\Big[\phi(z_n^{fb}) + z_n^{fb}\Phi(z_n^{fb})\Big] + \frac{\theta^2}{\gamma}z_n^{fb} - z_n^{fb}.
\end{aligned}
$$

Since $\frac{\delta^2}{\gamma}\Big[\phi(y) + y\Phi(y)\Big] + \frac{\theta^2}{\gamma}y - y$ decreases in y from Assumption (A1), it must be $z_n^* < z_n^{fb}$.

Therefore, for $z_n^* < z_n^{fb}$, it suffices to show $\lambda_n^* > z_n^*$, i.e.,

$$
\begin{aligned}
\frac{\delta^2}{\gamma}\Big[\phi(z_n^*) + z_n^*\Phi(z_n^*)\Big] + \frac{\theta^2}{\gamma}z_n^* &> z_n^* \\
\frac{\delta^2}{z_n^*}\phi(z_n^*) + \delta^2\Phi(z_n^*) + \theta^2 &> \gamma \\
\frac{\delta^2}{z_n^*}\phi(z_n^*) + \delta^2\Phi(z_n^*) &> \gamma - \theta^2 \qquad (24)
\end{aligned}
$$

If $\delta \to 0$, then $z_n^* \to 0$ and $\lambda_n^* \to 0$ from Eqs. (19)-(20). Note

$$
\begin{aligned}
H_{z_n^*} &\to -1, \\
H_{(\delta^2)} &\to \frac{1}{\theta^2}\left[\frac{\gamma}{\gamma - \theta^2}t_n^*\Phi(-t_n^*) - \phi(t_n^*)\right] = \frac{\phi(t_n^*)}{\theta^2}\left[\frac{\gamma}{\gamma - \theta^2}t_n^*\mu(t_n^*) - 1\right]
\end{aligned}
$$

Thus,

$$
\frac{\delta^2}{z_n^*} = \frac{d(\delta^2)}{dz_n^*} = -\frac{H_{z_n^*}}{H_{(\delta^2)}} \to \frac{1}{\phi(t_n^*)}\frac{\theta^2}{\frac{\gamma}{\gamma - \theta^2}t_n^*\mu(t_n^*) - 1}
$$

Then the left hand side of Eq. (24) becomes

$$\frac{\delta^2}{z_n^*}\phi(z_n^*) + \delta^2\Phi(z_n^*) \quad \to \quad \frac{\delta^2}{z_n^*}\phi(0) \to \frac{\phi(0)}{\phi(t_n^*)}\frac{\theta^2}{\frac{\gamma}{\gamma-\theta^2}t_n^*\mu(t_n^*)-1} > \frac{\theta^2}{\frac{\gamma}{\gamma-\theta^2}t_n^*\mu(t_n^*)-1}.$$

Suppose δ is sufficiently small. For Eq. (24) to hold, it suffices that

$$\begin{aligned}
\frac{\theta^2}{\frac{\gamma}{\gamma-\theta^2}t_n^*\mu(t_n^*)-1} &> \gamma-\theta^2 \\
\theta^2 &> \gamma t_n^*\mu(t_n^*) - (\gamma-\theta^2) \\
1 &> t_n^*\mu(t_n^*) = \frac{\gamma}{\theta^2}\mu(t_n^*)^2 \\
1 &> \frac{\gamma}{\theta^2}\mu(.75)^2 \approx \frac{\gamma}{\theta^2}0.56 \\
\frac{\theta^2}{0.56} > 1.5\theta^2 &> \gamma,
\end{aligned}$$

where the fourth inequality follows from $\mu' < 0$ and $t_n^* > 0.75$ if $z_n^* \to 0$ (as shown in the proof of Proposition 2); the last inequality follows from Assumption (A2).

Therefore, when δ is not very large, $z_n^* > z_n^{fb}$. Q.E.D.

Proof of Corollary 2. (i) From Proposition 2, z_n^* does not depend on σ. So, clearly $z^* = \sigma z_n^*$ increases in σ. From the proof of Lemma 1, $H' < 0$ and

$$\begin{aligned}
\frac{dH}{d(\theta^2)} &= \frac{\partial H}{\partial(\theta^2)} + \frac{\partial H}{\partial\lambda_n^*}\frac{d\lambda_n^*}{d(\theta^2)} + \frac{\partial H}{\partial t_n^*}\frac{dt_n^*}{d(\theta^2)} \\
&= \frac{1}{\gamma-\theta^2}\frac{\delta^2}{\theta^2}\left[\frac{\gamma}{\gamma-\theta^2}(t_n^*+\lambda_n^*)\Phi(z_n^*-t_n^*)\right] - \frac{z_n^*}{\theta^2} + \frac{\delta^2}{\theta^2}\frac{\gamma}{\gamma-\theta^2}\Phi(z_n^*-t_n^*)\frac{z_n^*}{\gamma} \\
&= \frac{1}{\gamma-\theta^2}\left[z_n^* + \frac{\delta^2}{\theta^2}\Big[\phi(z_n^*-t_n^*) + z_n^*\Phi(z_n^*-t_n^*)\Big]\right] - \frac{z_n^*}{\theta^2} + \frac{\delta^2}{\theta^2}\frac{\gamma}{\gamma-\theta^2}\Phi(z_n^*-t_n^*)\frac{z_n^*}{\gamma} \\
&> \frac{z_n^*}{\gamma-\theta^2} - \frac{z_n^*}{\theta^2} \\
&> 0
\end{aligned}$$

where the last inequality follows from Assumption (A2). Therefore,

$$\frac{\partial z_n^*}{\partial(\theta^2)} = -\frac{dH/d(\theta^2)}{dH/dz_n^*} > 0$$

so z_n^* increases in θ. Similarly z_n^* increases (decreases) in δ (γ). Since $z^* = \sigma z_n^*$, the same holds for z^*.

(ii) The comparative statics of k^* and x^* with respect to σ holds obviously for $t \leq t^*$. For $t > t^*$,

$$\frac{\partial k^*}{\partial \sigma} = \frac{\gamma\delta}{\gamma - \theta^2}\left[\lambda_n^* - \mu(\frac{t}{\sigma} - z_n^*) + \frac{t}{\sigma}\mu'(\frac{t}{\sigma} - z_n^*)\right].$$

Define $B(t) = \lambda_n^* - \mu(\frac{t}{\sigma} - z_n^*) + \frac{t}{\sigma}\mu'(\frac{t}{\sigma} - z_n^*)$. Since

$$\begin{aligned} B'(t) &= \frac{t}{\sigma^2}\mu''(\frac{t}{\sigma} - z_n^*) > 0, \\ B(t^*) &< -\frac{\theta^2}{\gamma}t_n^* + t_n^*\mu'(t_n^* - z_n^*) < 0. \end{aligned}$$

In addition,

$$\begin{aligned} \lim_{t\to\infty} B(t) &= \lim_{t\to\infty}\left[\lambda_n^* - \mu(\frac{t}{\sigma} - z_n^*) + z_n^*\mu'(\frac{t}{\sigma} - z_n^*) + (\frac{t}{\sigma} - z_n^*)\mu'(\frac{t}{\sigma} - z_n^*)\right] \\ &= \lambda_n^* - \lim_{y\to\infty}\mu(y) + z_n^*\lim_{y\to\infty}\mu'(y) + \lim_{y\to\infty} y\mu'(y) \\ &= \lambda_n^* > 0, \end{aligned}$$

where the last equality follows from the facts that $\lim_{y\to\infty}\mu(y) = \lim_{y\to\infty}\mu'(y) = \lim_{y\to\infty} y\mu'(y) = 0$.

Therefore, $\exists t_1 > t^*$ s.t. $\forall t \in (t^*, t_1)$, $B(t) < 0$; and $\forall t > t_1$, $B(t) > 0$. It follows that for $t \leq t_1$, $\partial k^*/\partial\sigma < 0$; and for $t > t_1$, $\partial k^*/\partial\sigma > 0$. Similarly we obtain the result for x^*.

For the other comparative statics, note that

$$\frac{\partial x^*(t)}{\partial \delta} = \frac{\sigma\theta}{\gamma-\theta^2}\left[\frac{t}{\sigma}+\frac{\gamma}{\theta^2}(\lambda_n^*-\mu(\frac{t}{\sigma}-z_n^*))+\frac{\delta\gamma}{\theta^2}(\frac{\partial \lambda_n^*}{\partial \delta}+\mu'(\frac{t}{\sigma}-z_n^*)\frac{\partial z_n^*}{\partial \delta})\right].$$

Using the facts that $\lim_{y\to\infty}\mu(y)=\lim_{y\to\infty}\mu'(y)=0$, and $\mu'<0$ and $\mu''>0$,

$$\begin{aligned}\lim_{t\to\infty}\frac{\partial x^*(t)}{\partial \delta} &> 0,\\ \frac{d}{dt}(\frac{\partial x^*(t)}{\partial \delta}) &= \frac{\sigma\theta}{\gamma-\theta^2}\Big[\frac{1}{\sigma}-\frac{\gamma}{\theta^2}\frac{1}{\sigma}\mu'(\frac{t}{\sigma}-z_n^*)+\frac{\delta\gamma}{\theta^2}\mu''(\frac{t}{\sigma}-z_n^*)\frac{1}{\sigma}\frac{\partial z_n^*}{\partial \delta}\Big] > 0.\end{aligned}$$

Therefore, x^* increases in δ when t is sufficiently high. Similarly, x^* increases (decreases) in θ (γ) when t is sufficiently high. The results about capital allocation can be obtained similarly. Q.E.D.

Proof of Proposition 3: From the proof of Proposition 2, it is clear that $b^*(t)=\frac{\gamma x^*}{\theta k^*}$ for $t\geq t^*$, and zero otherwise. The salary is

$$\begin{aligned}a^*(t) &= U(t)+0.5\gamma x^{*2}-b^*\Big[(\delta t+\theta x^*)k^*-0.5k^{*2}\Big]\\ &= \bar{U}+0.5\gamma x^{*2}+\int_{-\infty}^{t}\delta b^*k^*dt-b^*\Big[(\delta t+\theta x^*)k^*-0.5k^{*2}\Big]\\ &= \bar{U}+0.5\gamma x^*\frac{\theta b^*k^*}{\gamma}+\int_{-\infty}^{t}\delta b^*k^*dt-b^*k^*\Big[(\delta t+\theta x^*)-0.5k^*\Big]\\ &= \bar{U}+\int_{-\infty}^{t}\delta b^*k^*dt-b^*k^*\Big[0.5\delta t+0.5(\delta t+\theta x^*-k^*)\Big]\\ &= \bar{U}+\int_{-\infty}^{t}\delta b^*k^*dt-0.5\delta b^*k^*t,\end{aligned}$$

where the third equality follows from $\gamma x^*=\theta b^*k^*$; the last equality follows from the first order condition for k^*, $\delta t+\theta x^*-k^*=0$. Q.E.D.

Proof of Corollary 3. It is easy to verify that

$$b^{*\prime}(t) = -\frac{\lambda_n^*-\mu(\frac{t}{\sigma}-z_n^*)+\frac{t}{\sigma}\mu'(\frac{t}{\sigma}-z_n^*)}{\frac{\sigma\theta^2}{\gamma-\theta^2}\left[\frac{t}{\sigma}-\mu(\frac{t}{\sigma}-z_n^*)+\lambda_n^*\right]^2}$$

From the proof of Corollary 2, $\exists t_1 > t^*$ s.t. $\forall t \in (t^*, t_1)$, $B(t) < 0$; and $\forall t > t_1$, $B(t) > 0$, where $B(t)$ is the nominator in the above expression. Thus, $b^*(t)$ is increasing for $t \in (t^*, t_1)$ and decreasing for $t > t_1$.

Since $b^*(t)$ is a function of t/σ, its comparative statics with respect to σ is exactly the opposite of that with respect to t. Q.E.D.

References

[1] Antle, R., and G.D. Eppen, 1985, "Capital Rationing and Organizational Slack in Capital Budgeting," *Management Science* 31, 163-174.

[2] Berkovitch, E., and R. Israel, 2004, "Why the NPV Criterion does not Maximize NPV," *Review of Financial Studies* 17, 239-255.

[3] Bernardo, A.E., H. Cai, and J. Luo, 2001, "Capital Budgeting and Compensation with Asymmetric Information and Moral Hazard," *Journal of Financial Economics*, 61, 311-344.

[4] Christensen, Clayton M., 1997, The Innovator's Dilemma: When New Technologies Cause Great Firms to Fail 1997. Harper Collins, NY.

[5] Garcia, D., 2003, "Optimal Contracts with Privately Informed Agents and Active Principals," working paper, Dartmouth College.

[6] Gertner, R., D.S. Scharfstein, and J.C. Stein, 1994, "Internal versus External Capital Markets," *Quarterly Journal of Economics*, 109, 1211-1230.

[7] Gompers, Paul, Josh Lerner, and David S. Scharfstein, 2004, "Entrepreneurial Spawning: Public Corporations and the Genesis of New Ventures 1986 to 1999," forthcoming *Journal of Finance.*

[8] Harris, M., and A. Raviv, 1996, "The Capital Budgeting Process: Incentives and Information," *Journal of Finance*, 51, 1139-1174.

[9] Henderson, Rebecca, 1993, "Underinvestment and Incompetence as Responses to Radical Innovation: Evidence from the Photolithographic Alignment Equipment Industry," *Rand Journal of Economics*, 24, 248-270.

[10] Holmstrom, B., and P. Milgrom, 1987, "Aggregation and Linearity in the Provision of Intertemporal Incentives," *Econometrica*, 55, 597-619.

[11] Holmstrom, B., and J. Ricart i Costa, 1986, "Managerial Incentives and Capital Management," *Quarterly Journal of Economics*, 101, 835-860.

[12] Jovanovic, Boyan, 1979, "Job Matching and the Theory of Turnover, *Journal of Political Economy*, 87, 972-990.

[13] Laffont, J.-J., and J. Tirole, 1986, "Using Cost Observation to Regulate Firms," *Journal of Political Economy*, 94, 614-641.

[14] Laffont, J.-J., and J. Tirole, 1993, *A Theory of Incentives in Procurement and Regulation*, MIT Press, Cambridge.

[15] McAfee, R. P., and J. McMillan, 1987, "Competition for Agency Contracts," *RAND Journal of Economics*, 18, 296-307.

[16] Milgrom, P., and J. Roberts, 1992, *Economics, Organizations and Management*, Prentice Hall, Englewood Cliffs.

[17] Mirrlees, J.A., 1971, "An Exploration in the Theory of Optimum Income Taxation," *Review of Economic Studies*, 38, 135-208

[18] Mookherjee, D., and S. Reichelstein, 1992, "Dominant Strategy Implementation of Bayesian Incentive Compatible Allocation Rules," *Journal of Economic Theory*, 56, 378-99.

[19] Ozbas, O., 2004, "Integration, Organizational Processes and Allocation of Resources," *forthcoming Journal of Financial Economics.*

[20] Poterba, J., and L. Summers, 1992, "Time Horizons of American Firms: New Evidence from a Survey of CEOs," working paper, Harvard University.

[21] Prendergast, C., 1999, "The Provision of Incentives in Firms," *Journal of Economic Literature*, 37, 7-63.

[22] Prendergast, C., 2002, "The Tenuous Trade-off of Risk and Incentives?", *Journal of Political Economy*, 104, 1105-1134.

[23] Rajan, Raghuram, Henri Servaes and Luigi Zingales, 2000, "The Costs of Diversity: The Diversfication Discount and Inefficient Investment," *Journal of Finance*, 55, 35-80.

[24] Rosen, S., 1982, "Authority, Control, and the Distribution of Earnings," *Bell Journal of Economics*, 13, 311-321.

[25] Scharfstein, David S., and Jeremy Stein, 2000, "The Dark Side of Internal Capital Markets: Divisional Rent-seeking and Inefficient Investment," *Journal of Finance*, 55, 2537-2564.

[26] Shi, L., 2003, "Respondable Risk and Incentives: Evidence from Executive Compensation," working paper, University of Chicago.

[27] Stein, J.C., 1997, "Internal Capital Markets and the Competition for Corporate Resources," *Journal of Finance*, 52, 111-133.

[28] Szalay, D., 2003, "Procurement with an Endogenous Type Distribution," working paper, HEC.