# A no-arbitrage term structure model without latent factors

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This Draft: March 13, 2004

**NOTE: This paper is preliminary and subject to substantial change. Comments encouraged!**

**ABSTRACT**

I present a framework for modeling part of the dynamics of the term structure. The framework can be used to link the term structure to the dynamics of observed variables such as inflation and output. Its partial nature allows us to dispense with yield-based factors (e.g., latent factors) while retaining restrictions associated with no-arbitrage. I apply the model to the joint dynamics of inflation and the term structure. Both short-term and long-term bond yields change slowly after a change in inflation. I find that the dynamics of the price of interest rate risk needed to fit this pattern are implausible. An alternative interpretation of this pattern is that investors are systematically surprised by the slow adjustment of short rates to inflation.

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# 1 Introduction

Beginning with Vasicek (1977) and Cox, Ingersoll, and Ross (1985), researchers have built increasingly sophisticated no-arbitrage models of the term structure. These models specify the evolution of state variables under both the physical and equivalent martingale measures, and thus represent complete descriptions of the dynamic behavior of yields at all maturities. Much of this work focuses on latent variable settings, where the evolution of yields is described in terms of yields themselves. This rather introspective view was broadened by the important work of Piazzesi (2003) and Ang and Piazzesi (2003), who included macroeconomic variables in the workhorse latent variable framework of Duffie and Kan (1996). This extension allows us to investigate questions at the boundaries of macroeconomics and finance. For example, what does today's output gap tell us about the compensation investors demand to face interest rate risk? What does today's inflation rate tell us about the componenets of expected future real returns to nominal long-term bonds? These and related questions are now the focus of intensive research using models that describe the entire term structure with a combination of macroeconomic and latent factors.[1]

Yet many of these questions can be addressed without attempting to model the complete dynamics of the term structure. The logic follows the general asset-pricing approach introduced in Hansen and Singleton (1982), who noted that restrictions implied by no-arbitrage can be exploited without using (or knowing) the complete joint dynamics of asset prices and the pricing kernel. Recall that a zero-coupon bond's price is the expected value of the pricing kernel as of the time the bond matures. If we condition this expectation on the information in a specified set of macroeconomic variables and combine it with the dynamics of the same macro variables, we can exploit no-arbitrage restrictions without specifying the rest of the term structure. For example, the contemporaneous relation between a bond's price and the output gap is determined by the information in the output gap for both the expected time-path of short-term interest rates and expected time path of compensation investors require to face interest rate risk. In other words, it is determined by expectations conditioned on the output gap; other information is irrelevant.

In this paper I describe how to build and estimate partial term structure models that use Duffie-Kan dynamics, impose no-arbitrage, and do not include yield-based factors (whether they are latent or yields themselves). There are latent factors in the background, but they play no role in either the model's testable restrictions or in estimation. These partial models allow us to ask what the macroeconomic factors tell us about the future evolution of the term

[1] Recent work includes Dewachter, Lyrio, and Maes (2002), Dewachter and Lyrio (2002), Hördahl, Tristiani, and Vestin (2002), Ang and Bekaert (2003), Goto and Torous (2003), Ang, Piazzesi, and Wei (2003), and Rudebusch and Wu (2003).

structure. In particular, the covariation between bond yields and the observable factors can be decomposed into information about expected future short-term rates and information about expected future risk premia. The partial nature of the models means that certain questions cannot be addressed. For example, we cannot use them to ask what today's term structure tells us about the future evolution of the observable factors.

The tradeoffs between using a partial and a complete term structure model are standard. A complete model pins down more features of the data. If the features of the compete model that augment the partial model are correct, then exploiting these features increases estimation efficiency and ultimately tells us more about term structure dynamics. But if these features are wrong, imposing them can distort our view of the aspects of the model that we care about. In addition, the practical complications associated with term structure estimation are not trivial. In contrast to latent-factor models, the partial term structure model presented here can be estimated fairly easily.

I use this framework to investigate an underappreciated issue in the relation between inflation and the nominal term structure. It is well-known that in U.S. data, an increase in inflation corresponds to a subsequent increase in short-term rates for the next few quarters. The literature has also established a less-well-known fact: long-term rates respond to inflation in the same way. Therefore an increase in inflation corresponds to an immediate decrease in term premia (long-term yields less expected future short-term yields), which disappears over the next few quarters.

I find that the requirement of no-arbitrage, when coupled with these dynamics, implies highly unusual patterns in compensation for interest-rate risk. In the quarter that inflation increases, bond risk premia do not change substantially. However, investors expect that risk premia will fall sharply the next quarter, only to be reversed the quarter after that. This foreseeable, transitory variation in risk premia corresponds to wide variations in short-run expected excess returns to bonds. Although this pattern is a consequence of no-arbitrage, it is hard to give it an equilibrium interpretation. It is easier to tell a story in which investors are simply continually surprised by the slow adjustment of short rates to inflation. But since slow adjustment is a robust feature of U.S. data, investors would have to be remarkably uninformed about the joint dynamics of inflation and interest rates.

In the next section I describe the modeling framework. Section 3 describes the estimation methodology. Section 4 reviews the relevant facts about the relation between inflation and nominal bond yields. Section 5 presents results from estimation of a no-arbitrage model. Section 6 concludes.

# 2 A partial term structure model

Underlying the dynamics of bond yields is some structural model that explains these dynamics in terms of the state of the macroeconomy, central bank policy, and investors' willingness to bear interest-rate risk. Although the model here uses observable macro variables, it is not such a structural model. It is closer in spirit to a reduced-form model linking bond yields to macro variables.

## 2.1 The formal structure

Time is indexed by discrete periods $t, t+1, \ldots$. The length of a period is $h$ years. The period $t$ price and yield of a zero-coupon bond that pays a dollar at period $t+k$ are denoted $P_{k,t}$ and $y_{k,t}$. Yields are expressed as continuously-compounded annual rates. The short-term interest rate, which is equivalent to the yield on a one-period bond, is denoted $r_t$. An $n$-vector of macro variables is denoted $m_t^*$. The term "macro" is arbitrary here. In principle $m_t^*$ can include any observed variable that we are interested in relating to bond yields.

Project $r_t$ onto the history of $m_t^*$. To express this history in compact notation, stack lags zero through $p-1$ of $m_t^*$ in the vector $m_t$. The lag length $p$ must be chosen to capture all of the information in the history of $m_t^*$ about $r_t$. The projection is

$$r_t = \delta_0 + \delta_1' m_t + w_t. \tag{1}$$

The model's first assumption is to rule out a nonlinear relation between the short-term rate and $m_t$, so that the expectation of $w_t$ conditioned on $m_t$ is zero. Since the information in $m_t$ about $r_{t+k}$ is a subset of the information in $m_{t+k}$, conditional expectations of future $w$'s are also zero:

$$E(w_{t+k}|m_t) = E(E(w_{t+k}|m_{t+k})|m_t) = 0, \; k > 0. \tag{2}$$

The role of $w_t$ is to capture information in short-term rates that is not contained in $m_t$. Although the nature of the projection means that it is unforecastable with $m_t$, other information available to investors will allow them to forecast $w_t$. For example, the central bank may have recently decided to tighten monetary policy in response to events unrelated to $m_t$. If so, investors today expect this residual component to be greater than zero for some time. Investors may also have information about $m_{t+k}$ that is not contained in $m_t$. In particular, $w_t$ may contain information about $m_{t+k}$. For example, the exogeneous tightening of monetary policy may have implications for future realizations of the macro variables. The interpretation of (1) as a projection simply says that the predictability has to go in one direction: $w_t$ can predict future values of $m_t$, but $m_t$ cannot predict future values of $w_t$.

To complete the term structure model we need descriptions of the dynamics of $m_t$ and $w_t$ that are consistent with this one-way predictability. We also need to specify the dynamics of the pricing kernel. The discussion has to proceed carefully because at various points the model uses two different information sets: the complete set of information available to investors and the subset based on $m_t$. I use $E_t(\cdot)$ to represent expectations conditioned on the complete information set and $E(\cdot|m_t)$ to represent expectations conditioned on the more restrictive information set.

The model does not require a complete description of the dynamics of $m_t$. The relevant dynamics of $m_t$ are those conditioned on $m_t$. I assume that the one-step-ahead expectation of $m_t$ conditioned on $m_t$ is given by a VAR(1):

$$m_{t+1} = c + Fm_t + \epsilon_{t+1}, \quad E(\epsilon_{t+1}|m_t) = 0. \tag{3}$$

The same macro dynamics are used in Ang and Piazzesi (2003), although they assume these dynamics are based on investors' complete information sets. The linearity of the VAR is restrictive but the first-order dynamics are not because of the lags included in $m_t$. Owing to the first-order companion form, the parameters $c$, $F$, and $\Sigma$ can be expressed as

$$c = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}, \; F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \; F_2 = \begin{pmatrix} I & 0 \end{pmatrix},$$

where $c_1$ is a $n$-vector and $F_1$ is an $n \times pn$ matrix. The identity matrix in $F_2$ has $n(p-1)$ rows and the components represented with zeros are conformable. Note that we can write the $k$-ahead conditional forecast as

$$E\left(m_{t+k}|m_t\right) = (I - F)^{-1}(I - F^k)c + F^k m_t \tag{4}$$

where the identity matrices have the same dimensions as $F$.

Investors work with a more complete model that links the dynamics of $m_t$ to other variables in the economy. In contrast to (3), in this more complete model investors will distinguish among different kinds of shocks to $m_t$. For example, oil-price shocks may die out at a different rate than a policy shock. This more restrictive model should be viewed as the result of integrating out all non-$m_t$ information from this more complete model. In particular, in the model used by investors the "shock" $\epsilon_{t+1}$ need not be entirely unexpected. The subscript on this innovation indicates that it enters the restricted information set at $t+1$. However, the expectation of $\epsilon_{t+1}$ conditioned on investors' complete information set at time $t$ is not necessarily zero. Investors may have information about the evolution of macro variables that is not contained in lagged macro variables. The one restriction that is needed on investors' complete information set is that the uncertainty in $\epsilon_{t+1}$ is normally distributed:

$$\epsilon_{t+1} - E_t(\epsilon_{t+1}) \sim N(0, \Sigma), \quad \Sigma = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

where $\Sigma_{11}$ is an $n \times n$ positive definite matrix. (The zeros in $\Sigma$ are a consequence of the companion form.)

Although the non-macro component of the short-term rate $w_t$ drops out of the testable part of the model, we need to put some structure on its dynamics in order to justify dropping it. I assume that $w_t$ can be written as an linear function of some latent state vector $x_t$ of arbitrary length:

$$w_t = \delta_2' x_t. \tag{6}$$

This state vector is observed by investors. Its data-generating process is affine conditional on investors' complete information set:

$$x_{t+1} = F_x x_t + \sigma(x_t)\eta_{t+1}, \tag{7}$$

$$\sigma(x_t)\sigma(x_t)' = v_0 + v_1 x_t, \tag{8}$$

$$\eta_{t+1} \sim N(0, I), \ \mathrm{Cov}_t(\eta_{t+1}, \epsilon_{t+1}) = 0. \tag{9}$$

This structure implies that the expectation of $x_t$ conditional on $m_t$ is zero, as required by interpretation of $w_t$ as the residual from a projection onto $m_t$. The vector $v_0$ and matrices $F_x$ and $v_1$ are free parameters but play no role in the model's implications.

Nothing in these dynamics rules out a correlation between $x_t$ and investors' time-$t$ expectations of $\epsilon_{t+k}, k > 0$. A concrete example helps to interpret these dynamics. Let the observable factor $m_t^*$ consist of a single variable: the CPI inflation rate from $t-1$ to $t$. Imagine that at the end of period $t$ OPEC announces that it will cut sales of crude oil. This will raise producer prices immediately but will not affect the period-$t$ CPI. The announcement raises investors' expectation of inflation in $t+1$, thus the nominal short-term rate will rise to compensate investors. Hence $r_t$ will be higher than investors expected as of the end of quarter $t-1$. Because this innovation to $r_t$ is unrelated to the observed history of the CPI inflation rate, it shows up as a shock to latent factors $\eta_t$. This shock is correlated with next period's innovation to the CPI inflation rate $\epsilon_{t+1}$. If oil price shocks are completely incorporated into the CPI within a period, the shock $\eta_t$ will not persist. In other words, by the end of $t+1$ the OPEC action will be embedded in $m_{t+1}$, not in the residual component $w_{t+1}$.

No-arbitrage implies the existence of a pricing kernel $z_{t+1}$ such that $P_{k,t} = E_t(z_{t+1}P_{k-1,t+1})$ for all maturities $k$. The key assumption about the pricing kernel is that the compensation investors require to face latent-factor risk depends only on the latent factors. The dynamics of the pricing kernel conditional on investors' complete information set are

$$z_{t+1} = \exp\left[-hr_t - \lambda_t'(\epsilon_{t+1} - E_t(\epsilon_{t+1})) - d(x_t)'\eta_{t+1} - \frac{1}{2}\mathrm{Var}_t(\lambda_t'\epsilon_{t+1} + d(x_t)'\eta_{t+1})\right], \quad (10)$$

$$\sigma(x_t)d(x_t) = s_0 + s_1 x_t, \quad (11)$$

$$\lambda_t = \lambda_0 + \lambda_1 m_t + \lambda_2 x_t, \quad (12)$$

where the free parameters in $\lambda_1$ are in an $n \times np$ matrix $\lambda$:

$$\lambda_1 = \begin{pmatrix} \Sigma_{11}^{-1}\lambda \\ 0 \end{pmatrix}.$$

The zeros in $\lambda_1$ are not a restriction because all but the first $n$ elements of $\epsilon_{t+1}$ are zero. The vectors $s_0$ and $\lambda_0$ and the matrices $s_1$ and $\lambda_2$ are free parameters but they play no role in the model's implications.

These assumptions allow us to use standard techniques to solve for bond prices. Assume that log bond prices are affine in $m_t$ and $x_t$:

$$\log P_{k,t} = A_k + B_{1,k}' m_t + B_{2,k}' x_t. \quad (13)$$

Because log bond prices and the pricing kernel are both normally distributed conditioned on investors' complete information sets, we can rewrite the no-arbitrage condition $P_{k,t} = E_t(z_{t+1}P_{k-1,t+1})$ as

$$\log P_{k,t} = -hr_t + E_t(\log P_{k-1,t+1}) + \frac{1}{2}\mathrm{Var}_t(\log P_{k-1,t+1}) + \mathrm{Cov}_t(\log P_{k-1,t+1}, z_{t+1}). \quad (14)$$

Plugging (13) into (14) and using (5), (7), (8), (9), and (10) produces

$$\begin{aligned} A_k + B_{1,k}' m_t + B_{2,k}' x_t \;&=\; -hr_t + A_{k-1} + B_{1,k-1}' E_t(m_{t+1}) + B_{2,k-1}' E_t(x_{t+1}) \\ &\quad + \frac{1}{2}\left(B_{1,k-1}'\Sigma B_{1,k-1} + B_{2,k-1}'\sigma(x_t)\sigma(x_t)' B_{2,k-1}\right) \\ &\quad - B_{1,k-1}'\Sigma\lambda_t - B_{2,k-1}'\sigma(x_t)d(x_t). \end{aligned} \quad (15)$$

Take the expectation of (15) conditioned on $m_t$ and evaluate it using (3), (8), and (11):

$$\begin{aligned} A_k + B'_{1,k} m_t &= -h(\delta_0 + \delta'_1 m_t) + A_{k-1} + B'_{1,k-1}(c + F m_t) \\ &\quad + \frac{1}{2}\left(B'_{1,k-1}\Sigma B_{1,k-1} + B'_{2,k-1} v_0 B_{2,k-1}\right) \\ &\quad - B'_{1,k-1}\Sigma(\lambda_0 + \lambda_1 m_t) - B'_{2,k-1} s_0. \end{aligned} \tag{16}$$

Matching coefficients on $m_t$ produces a recursive equation that defines $B_{1,k}$.

$$B'_{1,k} = -h\delta'_1 + B'_{1,k-1}F - B_{1,k-1}\Sigma\lambda_1$$

The solution is

$$B'_{1,k} = -h\delta'_1 (I - \tilde{F})^{-1}(I - \tilde{F}^k) \tag{17}$$

where $\tilde{F}$ is the equivalent-martingale counterpart to $F$:

$$\tilde{F} = \begin{pmatrix} F_1 - \lambda \\ F_2 \end{pmatrix}. \tag{18}$$

## 2.2 Interpreting the model

Bond yields covary with the macro factors because the macro factors are correlated with the expected path of the short-term interest rate. If investors' attitudes toward risk did not depend on the macro factors, the dynamics of the macro factors $F$ and the sensivitity of the short rate to these factors $\delta_1$ would determine the responsiveness of a bond's yield to these factors. In other words, the physical dynamics $F$ would be the same as the equivalent-martingale dynamics $\tilde{F}$. The model allows the price of macro risk to depend on the level of the macro variables, which drives a wedge between the physical and equivalent-martingale dynamics.

A critical assumption underlying (17) is that the compensation investors demand to face latent-factor risk does not depend on $m_t$. To understand the role of this restriction, consider replacing (11) with the more general form

$$\sigma(m_t, x_t) d(m_t, x_t) = s_{20} + s_{21} m_t + s_{22} x_t.$$

This general form allows both the conditional variance of the latent factors and the price of $\eta_t$ risk to depend on $m_t$. With this form, the testable restrictions implied by the partial term structure model disappear. The conditional covariance between the log bond price and the log pricing kernel is then

$$\text{Cov}_t(\log P_{k-1,t+1}, \log z_{t+1}) = -B'_{1,k-1}\Sigma(\lambda_0 + \lambda_1 m_t + \lambda_2 x_t) - B'_{2,k-1}(s_{20} + s_{21} m_t + s_{22} x_t)$$

and its expectation conditioned on $m_t$ is

$$E_t\left[\text{Cov}_t(\log P_{k-1,t+1}, \log z_{t+1})\right] = -B'_{1,k-1}\Sigma(\lambda_0 + \lambda_1 m_t) - B'_{2,k-1}(s_{20} + s_{21} m_t).$$

Replace the last line of (16) with this expected covariance and match coefficients in $m_t$. The result is a a recursive equation for $B_{1,k}$ that depends on $B_{2,k}$. Therefore we need a solution to the entire term structure model in order to calculate $B_{1,k}$.

Put differently, the restrictions of no-arbitrage are driven by a comparison of risk and expected excess returns across assets. The factor loadings $B_{1,k}$ give us a relative measure of $\epsilon_t$ risk across bonds, but we have no relative measure of $\eta_t$ risk across bonds without a parameterized description of the dynamics of $x_t$. Therefore if $m_t$ affects the price of $\eta_t$ risk, no-arbitrage does not allow us to say much about the relation between $m_t$ and the term structure.[2]

It is hard to defend this assumption except on the grounds of practicality. This is especially true given that shocks to the macro variables and shocks to the latent factors can be qualitatively similar shocks. To fix ideas, let $m_t^*$ be CPI inflation. Consider two different shocks conditional on investors' information as of period $t$. The first is a positive shock to inflation in period $t+1$. The second is a policy shock realized in period $t+1$ that will raise inflation beginning in $t+2$. In the model, the price of risk of the former shock is allowed to depend on current inflation. Because the policy shock will show up as a latent factor shock in $t+1$, its price of risk is not allowed to depend on current inflation.

Other assumptions in the model are less important. In particular, the affine form for $x_t$ is not essential. The only reason the affine form is used is to guarantee joint log-normality of bond prices and the pricing kernel. Log-normality implies (14), which is the equation that allows us to use (15) to match up the terms in $m_t$. A more general representation replaces the linear form of the residual (6) with

$$w_t = f_1(x_t), \quad E(f_1(x_t)|m_t) = 0.$$

Log-normality requires that conditional on investors' complete information set, innovations

[2] The same point applies to the variance-covariance matrix of latent variable shocks. If this variance depends on $m_t$, a Jensen's inequality term in (15) will not disappear after conditioning on $m_t$.

in $f_1(x_t)$ are normal:

$$f_1(x_{t+1}) - E_t(f_1(x_{t+1})) \sim N(0, V_1(x_t)).$$

Similarly, we can replace the affine form of (13) with

$$\log P_{k,t} = A_k + B'_{1,k} m_t + f_k(x_t)$$

where again log-normality requires

$$f_k(x_{t+1}) - E_t(f_k(x_{t+1})) \sim N(0, V_k(x_t)).$$

The affine dynamics of $x_t$ results in functional forms of $f_k(x_t)$ that are consistent with the requirement of log-normality. But since the relation between $x_t$ and the term structure is not of direct interest, we might just as well simply assume log-normality to motivate the model's restrictions involving $m_t$.

## 2.3 Applications

The model links $m_t$ to the dynamics of the term structure through the vector $c$ and the matrix $F$ of the macroeconomic VAR(1), the vector $\delta_1$ linking the short rate to $m_t$, and the matrix $\lambda$ that describes the compensation investors demand to face $m_t$ risk. The remainder of the term structure is not modeled. With this partial model of the term structure we can ask the following kinds of questions.

1. How does the expected time-path of $r_t$ depend on $m_t$?

   The response of the short rate to a macroeconomic shock $\epsilon_t$ is $\delta_1' \epsilon_t$. The expected change in the short rate from $t$ to $t+k$, conditioned on $m_t$, is

   $$E(r_{t+k} - r_t | m_t) = \delta_1' \left[ (I - F^k)c - (I - F^k) m_t \right]. \tag{19}$$

   The final term in square brackets uses (4). One application of these expressions is to investigate Taylor-rule descriptions of the short rate. Note that this $k$-ahead forecast is not a minimum-variance forecast. There is additional information in the term structure (at a minimum, the current level of the short rate) that is ignored in forming this conditional expectation. Therefore this model should not be used as a forecasting tool, but rather as a way to link the macroeconomic variables with the term structure.

2. How do risk premia on bonds vary with $m_t$?

The partial nature of the model does not allow us to say anything about mean excess returns to bonds. However, it does allow us to determine how variations in $m_t$ correspond to variations in expected excess returns. The expected log return to a $k$-maturity bond held from $t$ to $t+1$, conditioned on $m_t$, is

$$E(\log P_{k-1,t+1}|m_t) - \log P_{k,t} = hr_t + \kappa_{k-1} + B'_{1,k-1}\lambda m_t.$$

The constant term $\kappa_{k-1}$ is unrestricted.

3. How does the term structure react to an innovation in a macroeconomic variable?

   Consider the expectation of the $k$-maturity yield conditional on the contemporaneous $m_t$. (In other words, we observe $m_t$ but we have no observations of $x_{t-s}, s \geq 0$.) The expectation is

$$y_{k,t} = E(y_{k,t}|m_t) + \nu_{k,t}, \tag{20}$$

$$E(y_{k,t}|m_t) = a_k + (1/k)\delta_1'(I - \tilde{F})^{-1}(I - \tilde{F}^k)m_t, \tag{21}$$

$$a_k = -(1/k)A_k, \quad \nu_{k,t} = -(1/k)B'_{2,k}x_t. \tag{22}$$

   The constant term $a_k$ is unrestricted. The effect on $y_{k,t}$ of the macroeconomic shock $\epsilon_t$ is $(1/k)\delta_1'(I - \tilde{F})^{-1}(I - \tilde{F}^k)\epsilon_t$.

4. What does $m_t$ tell us about the future evolution of the term structure?

   The $k$-period-ahead forecast of the change in the yield on a bond with constant maturity $q$ is

$$\begin{aligned} &E(y_{q,t+k} - y_{q,t}|m_t) = \\ &\frac{1}{q}\delta_1'(I - \tilde{F})^{-1}(I - \tilde{F}^q)\left[(I - F)^{-1}(I - F^k)c - (I - F^k)m_t\right]. \end{aligned} \tag{23}$$

5. Is the empirical failure of the expectations hypothesis associated with $m_t$?

   Campbell and Shiller (1991) estimated regressions of the form

$$y_{l-s,t+s} - y_{l,t} = b_0 + b_1\frac{s}{l-s}(y_{l,t} - y_{s,t}) + e_{t+s,l,s} \tag{24}$$

   for maturities $l > s$. Under the weak form of the expectations hypothesis the coefficient $b_1$ should equal one, but in the data it is often negative. A common interpretation of this result is that bond risk premia and the slope of the term structure are positively correlated. Is this also true of the variability in the term structure that is associated with variations in $m_t$? Consider estimating (24) using $m_t$ as instruments. The conditional expectation of yield spread on the right of (24) can be expressed as

$$E(y_{l,t} - y_{s,t}|m_t) = \rho_{l,s} + \left(-\frac{1}{l}B_{1,l} + \frac{1}{s}B_{1,s}\right)' m_t$$

where $\rho_{l,s}$ is an unrestricted constant. The conditional expectation of the left side of (24) can be expressed as

$$E(y_{l-s,t+s} - y_{l,t}|m_t) = \phi_{l,s} + \frac{s}{l-s}E(y_{l,t} - y_{s,t}|m_t) - \frac{1}{l-s}B'_{1,l-s}\left(F^s - \tilde{F}^s\right)m_t \quad (25)$$

where $\phi_{l,s}$ is an unrestricted constant. If $\lambda = 0$, then $F = \tilde{F}$ and the final term in (25) is identically zero. In this case, the population estimate of $b_1$ from IV estimation of (24) is one. More generally, the population regression coefficient is

$$\begin{aligned}\hat{b}_1 &= 1 - \frac{1}{s}\left[\left(-\frac{1}{l}B_{1,l} + \frac{1}{s}B_{1,s}\right)' \mathrm{Var}(m)\left(-\frac{1}{l}B_{1,l} + \frac{1}{s}B_{1,s}\right)\right]^{-1} \times \\ &\quad \left(-\frac{1}{l}B_{1,l} + \frac{1}{s}B_{1,s}\right)' \mathrm{Var}(m)\left(F^s - \tilde{F}^s\right)' B_{1,l-s}\end{aligned}$$

where $\mathrm{Var}(m)$ is the unconditional variance-covariance matrix of $m_t$. Given this variance and the parameters of the term structure model, the regression coefficient can be computed.

# 3 Econometric methodology

I consider a setting in which the econometrician observes, at each date $1, 2, \ldots, T$, the macroeconomic vector $m_t^*$ and yields on $J$ bonds with constant maturities $k_j, j = 1, \ldots, J$. The natural technique to estimate the parameters of the model is the Generalized Method of Moments (GMM) of Hansen (1982). One obvious set of moments are the OLS moment conditions associated with the macroeconomic vector autoregression. The $n(1 + np)$ moments associated with observation $t$ are

$$[m_t^* - c_1 - F_1 m_{t-1}] \otimes (1 \ \ m'_{t-1})'. \quad (26)$$

The choice of moment conditions to identify the term structure parameters is not as clear. One possibility is to use (20) and (21). More precisely, we can introduce additional free parameters $a_1$ through $a_J$ and use the following $np + 1$ moments associated with bond

$j$ at time $t$:

$$\left[y_{k_j,t} - a_j - \frac{-1}{k_j} B'_{1,k_j} m_t\right] \otimes (1 \;\; m_t')'. \tag{27}$$

Unfortunately, typical choices of macroeconomic variables produce residuals $\nu_t$ in (20) that are highly serially correlated. For example, the residuals from Taylor-rule regressions of the short-term interest rate are close to a random walk.[3] Although this behavior does not invalidate the moment conditions, it creates substantial problems for statistical inference.

Both yield spreads and first-differences of yields are less persistent than are yields. We can use the following $np$ moments for the first difference of the $k_j$-maturity yield:

$$\left[(y_{k_j,t} - y_{k_j,t-1}) - \frac{-1}{k_j} B'_{1,k_j}(m_t - m_{t-1})\right] \otimes (m_t - m_{t-1}). \tag{28}$$

Another advantage of (28) relative to (27) is that constant terms do not need to be added to the set of free parameters. If there is a nonzero correlation between $x_t$ and $m_{t+s}, s > 0$, this moment condition may not have expectation zero because of the possibility that $m_t$ is correlated with the residual of the bond yield observed at $t-1$. In this case the right side of the Kronecker product can be replaced with instruments dated $t-1$ or earlier, such as $m_{t-1}$.

An $np+1$ moment vector for the yield spread between the $k_j$-maturity bond and the $k_k$-maturity bond is

$$\left[(y_{k_j,t} - y_{k_k,t}) - a_{j,k} - \left(\frac{-1}{k_j} B'_{1,k_j} - \frac{-1}{k_k} B'_{1,k_k}\right) m_t\right] \otimes (1 \;\; m_t')' \tag{29}$$

This moment condition requires the additional free parameter $a_{j,k}$.

In practice I use a combination of (26) and (28). The model's parameters are exactly identified if there is the number of observed bond yields is one greater than the number of macroeconomic factors. Additional bond yields allow overidentifying tests of the model.

We can also test for restrictions on the dynamics of the price of risk. There are two special cases of interest. The first is when the compensation required by investors to face macro risk depends only on the current value of the macroeconomic variables. In this case we can write $\lambda$ as

$$\lambda = (\lambda_{11} \mid 0)$$

where $\lambda_{11}$ is an $n$ by $n$ matrix. Another special case is when compensation depends only on

[3]See, e.g., Rudebusch (2002).

the one-step-ahead forecast of the macroeconomic variables. In this case we can write $\lambda$ as

$$\lambda = (\lambda_{11} \mid 0)\, F$$

Both of these cases impose $(p-1)n$ additional restrictions on the parameter vector. (Neither of these special cases is meaningful if the number of lags $p$ is one.)

## 4 Inflation and the term structure

Short-term interest rates react slowly to inflation. (By reaction, I refer to a statistical relation. Causation is a big issue that is not central to the discussion here.) Most of the relevant evidence is found in the literature on the Taylor rule. The description of the central bank's reaction function in Taylor (1993) is that the current Fed funds rate is based in part on inflation over the past year. Put differently, current short-term rates depend on lags zero through three of quarterly inflation. There is an ongoing debate over whether the Fed reacts more slowly to inflation (and output) than is implied by this rule. Clarida, Galí, and Gertler (2000) recommend adjusting the Taylor rule to account for the the desire of the Fed to smooth rates. Rudebusch (2002) argues that the evidence is consistent with the Taylor rule combined with persistent polich shocks, while English, Nelson, and Sack (2003) claim that a better description of the data is slow reaction to inflation and output combined with persistent polich shocks. A recent review of the evidence is in Sack and Wieland (2000).

The literature on macroeconomic vector autoregressions contains similar evidence, although the primary focus of that literature is on identifying the effects of monetary policy shocks. For example, Leeper, Sims, and Zha (1996) contains some results on the responsiveness of short-term rates to inflation shocks. A related literature beginning with Fama (1975) reverses the question of predictability by considering the forecast power of interest rates for future inflation. This direction of predictability cannot be addressed by the partial term structure model developed in this paper, thus I do not consider it further.

Research on the Taylor rule, as well as macro vector autoregressions, typically concentrates on the dynamics of short-term rates rather than the dynamics of long-term rates. An exception in the Taylor rule literature is Rudebusch, who uses information in forward rates combined with the assumption that risk premia do not depend on the level of short-term rates. Exceptions in the vector autoregression literature are Bernanke, Gertler, and Watson (1997), Evans and Marshall (1998), and Evans and Marshall (2002). They examine how both short-term and long-term yields respond to various shocks. The vector autoregression structure allows them to construct expected short-term interest rates, which in turn allows them to discuss the response of term premia (long-term yields less expected short-term yields) to these shocks.[4] Ang and Piazzesi (2003) also reports VAR evidence on the reaction of bond yields to inflation. A broad conclusion from this work is that both short-term and long-term yields adjust slowly to inflation shocks.

Some regression results relating short-term rates, long-term rates, and inflation will illustrate the nature of the slow adjustment and the puzzle that it raises. The data I use are quarterly from 1953Q1 through 2003Q4. The short-term rate is the three-month yield from the Center for Research in Security Prices (CRSP) riskfree rate file. The long-term rate is the five-year zero-coupon yield from the CRSP Fama-Bliss file. Both yields are measured on the last day in the quarter. Inflation is measured by the quarterly log change in the personal consumption expenditure (PCE) chained price index. All data are continuously compounded and expressed as annual rates. Note that inflation is measured by quarterly changes in log prices, not four-quarter changes in log prices.

One way to analyze the joint dynamics of these variables is to estimate a vector autoregression. For ease of comparison with the term structure model estimated in the next section, I instead estimate a univariate AR(4) for inflation and link these dynamics to bond yields by estimating regressions of yields on lags zero through three of inflation.

$$\pi_t = b_{\pi,0} + \sum_{i=1}^{4} b_{\pi,i}\pi_{t-i} + e_{\pi,t} \tag{30}$$

$$y_{3m,t} = b_{3m,0} + \sum_{i=0}^{3} b_{3m,1+i}\pi_{t-i} + u_{3m,t}, \;\; u_{3m,t} = \rho_{3m}u_{3m,t-1} + e_{3m,t} \tag{31}$$

$$y_{5y,t} = b_{5y,0} + \sum_{i=0}^{3} b_{5y,1+i}\pi_{t-i} + u_{5y,t}, \;\; u_{5y,t} = \rho_{5y}u_{5y,t-1} + e_{5y,t} \tag{32}$$

The AR(1) structure of the residuals in (31) and (32) picks up their high serial correlation (around 0.9 in quarterly data). Regression (31) differs from a standard Taylor rule regression in two ways. First, the coefficients on quarterly inflation are allowed to differ from each other. The standard regression uses annual inflation, which is equivalent to requiring constant coefficients on quarterly inflation. Second, no measure of the output gap is included. Including such a measure (and its associated dynamics) makes it harder to isolate the links between inflation and the term structure. I discuss this issue in more detail in the next section.

[4] Diebold, Rudebusch, and Aruoba (2003) also use vector autoregressions to jointly model the yield curve and macroeconomic variables, but they do not consider the contemporaneous effect of inflation shocks on the term structure.

The objective here is to give a snapshot of the data dynamics that are analyzed more formally in the next section. Therefore I do not report individual parameter estimates or standard errors. Figure 1 displays the relevant patterns in the data. The solid line in Panel A is the effect of a unit shock to inflation on current and future inflation implied by the results of (30). The combination of this impulse response and the results of (31) can be used to determine the expected time path of the response of the three-month yield given to the shock. The dashed line in Panel A displays this path. The short rate immediately moves in the same direction as the inflation shock, increasing by about 10 percent of the shock. Within two more quarters the short rate has jumped to about 40 percent of the inflation shock.

Given this expected path of the three-month yield, we can determine the effect of the inflation shock on the average three-month yield over the next five years. This corresponds to the change in the five-year yield under the assumption that current and future risk premia are unaffected by the inflation shock. The solid line in Panel B displays the impulse response of this theoretical five-year yield. The dashed line in Panel B is the impulse response of the actual five-year yield, computed using the results of (32).

The puzzle is clear from Panel B. The immediate reaction of the long-term bond yield is much less than the forecasted reaction within two quarters. For example, when inflation jumps up by one percent, investors are willing to accept a modest increase in the long-term bond yield of five basis points even though the long-term yield is expected to rise by another fifteen basis points in the near future.

One potential criticism of this analysis is that inflation data are not available on a real-time basis. Thus the term structure as of the end of quarter $t$ does not necessarily fully reflect the information in $\pi_t$. Since the delayed reaction on the part of bond yields is concentrated in quarter $t+2$, not quarter $t+1$, this is almost certainly not a valid criticism. Nonetheless, to eliminate this as a possibility I recalculated the results using a different measure of inflation. The alternative measure of inflation in quarter $t$ is the log change in the Consumer Price Index from the middle month of quarter $t-1$ to the middle month of quarter $t$. This measure is known by the end of the last month of quarter $t$. The results, which I do not report in detail, are very close to the results using the PCE price index.

There are two different interpretations of Fig. 1. The first interpretation is that the estimated dynamics are accurate, investors know these dynamics, and the curious behavior of the long-term yield is produced by complicated dynamics of the price of interest rate risk. This interpretation is the focus of the empirical work in the next section. The second interpretation is that dynamics plotted in Fig. 1 do not correspond to investors' perceptions of the dynamics. This can occur either because investors are irrational or because the sample dynamics over the past fifty years are not representative of the data-generating process.

Although I do not formally investigate the second interpretation, suggestive evidence is provided by estimating the regressions separately over the two subsamples 1953Q1 through 1982Q4 and 1983Q1 through 2003Q4. The relevant dynamics are plotted in Figure 2 for the early sample and in Figure 3 for the later sample. Although the dynamics differ in many ways across the two samples, the main puzzle holds: The immediate response of the long-term yield is much less than the response within two quarters. Thus this feature appears to be a robust feature of the data; if investors are unaware of it, they are ignoring a well-established pattern. Yet the responses of the short rate to inflation (the dashed lines in the Panel A's) and the responses of the long-term yield to inflation (the dashed lines in the Panel B's) are so similar that it is hard to discount the view that investors are simply continually surprised by the responses of the short rate.

# 5 Estimation results

In this section I use the framework of Section 2 to formally investigate the relation between inflation and the term structure. The vector of observed macroeconomic variables is

$$m_t = \begin{pmatrix} \pi_t & \pi_{t-1} & \pi_{t-2} & \pi_{t-3} \end{pmatrix}$$

where $\pi_t$ is the PCE inflation rate discussed in the previous section. At the end of each quarter, yields on four zero-coupon securities are observed. Their constant maturities are three months, one year, three years, and five years. Yield data are from CRSP.

There are 13 parameters to estimate. Five of these correspond to the AR(4) description of inflation from (3). They are the scalar $c$ and the four-element vector $F_1$. There are also four elements of the vector $\delta_1$ in (1). (The scalar $\delta_0$ in this equation could also be estimated but plays no role in the model.) Finally, there are four elements in the vector $\lambda$ that represents the difference between the physical and equivalent martingale AR(4) dynamics of inflation in (18).

An alternative approach is to use the same framework but augment the macroeconomic variables with some measure of the output gap. The resulting model would then look like the model of Ang and Piazzesi (2003) without the latent variables (and with quarterly instead of monthly dynamics). The main problem with this expanded setup is that the large number of free parameters make it difficult to interpret the results. The expanded model has 42 free parameters. A nice feature of the term structure framework of this paper is that we do not require that the variables in $m_t$ capture all, or even much, of the dynamics of the term structure. Therefore we do not need to include information about the output gap in order to model the relation between inflation and the term structure.

Estimation is with GMM. I use 21 moment conditions. Five are moment conditions of the AR(4) description of inflation from (26). For each of the four Treasury securities there are an additional four moment conditions from (28) that represent the ability of differenced inflation to explain differenced yields.[5] Stack the moment conditions for quarter $t$ in the vector $f_t$. For $T$ observations of this vector, the parameter estimates solve the problem

$$\min J = T g_T' W g_T, \quad g_T = \frac{1}{T}\sum_{i=1}^{T} f_t, \tag{33}$$

for some weighting matrix $W$.

I use two iterations of GMM. For the first iteration, the weighting matrix is the inverse of the sample covariance matrix of the moments evaluated at "OLS/risk-neutral" parameters. This means that the parameters $c$ and $F_1$ are given by OLS estimation of the AR(4) for inflation, the parameter $\delta_1$ vector is given by a regression of the three-month yield on lags zero through four of inflation, and $\lambda$ is set to zero. The resulting GMM parameter estimates are then used to construct an asymptotically efficient weighting matrix and the parameters are estimated again. The covariance matrices are estimated using the robust method of Newey and West (1987) with four moving average lags to account for the negative serial correlation in first differences of yields.

The solution to (33) requires nonlinear optimization. To find the global minimum, I randomly generated 50 starting values. For each starting value, I used Simplex (IMSL routine dumpol) to get in a well-behaved neighborhood of the parameter estimates. I then used a derivative-based algorithm (IMSL routine dbconf) to improve the accuracy of the estimates.

Table 1 displays the results of estimating the unrestricted model over the sample period 1953Q1 through 2003Q4. Panel A reports the parameter estimates and Panel B reports overidentifying tests of the model. The parameter estimates of $\delta_1$ and $F_1$ are unsurprising. In particular, the short-term interest rate reacts to both current and lagged inflation. The parameters of the price of risk, $\lambda$, appear to reconcile this slow adjustment of the short rate to the slow adjustment of long-term bond yields. The $\chi^2(8)$ test of the adequacy of the model (the first row in Panel B) does not come close to rejecting the model. Visual evidence

[5] I experimented with the instrumental variables version of this moment condition, as discussed in the text after (28). When $m_t$ is used as the instrument vector, the model's parameter estimates do not correspond to our intuition about the values of these parameters. In particular, the elements of $\delta_1$ are generally negative, indicating that short-term rates fall when inflation rises. I therefore report results using (28) instead of its IV version.

of the success of the model is in Panel A of Figure 4. The dashed line in the figure displays the actual response of the five-year yield and the solid line displays the response implied by the model's parameter estimates. The basic pattern is reproduced by the model. Without the flexibility provided by $\lambda$, the model cannot reproduce this behavior. The second row of Panel B reports that the restriction $\lambda = 0$ is overwhelmingly rejected by the likelihood ratio test of Newey and West (1987).

Yet the way that the model fits the data is economically unsatisfactory. Expected returns to long-term bonds fluctuate dramatically in response to inflation. Visual evidence is in Panel B of Fig. 4. The solid line represents the response of the expected quarterly return to three-month bonds. (This is simply the yield on the bond divided by four.) The shape of this line corresponds to the slow adjustment of three-month yields to inflation. The dashed line represents the response of the expected quarterly return to a five-year bond. The difference between the two lines is the expected excess return to the five-year bond. According to the model, this expected excess return oscillates sharply from quarter to quarter.

For concreteness, consider the implications of this figure given a quarter-$t$ increase of one percentage point in the inflation rate. As of the end of quarter $t$, the return to holding a three-month bond through the end of quarter $t+1$ is higher by about three basis points (twelve basis points in annualized yield). The expected return to holding a five-year bond from the end of quarter $t$ to the end of quarter $t+1$ is higher by about eleven basis points. One quarter later, the return to the three-month bond is roughly unchanged from quarter $t$, while the expected return to the five-year bond is nearly 60 basis points below its quarter-$t$ value. After two more quarters, the expected return to the five-year bond has again reversed course.

Expected long-term bond returns behave this way because the maintained hypothesis of the model is that investors foresee the sharp increase in long-term bond yields from $t+1$ to $t+2$. The model fits these dynamics through a sharp, transitory change in the price of risk that kicks in one quarter after the change in inflation. More precisely, the second element of $\lambda$ is large, while the sum of the coefficients in $\lambda$ is close to zero. Although there is always some general equilibrium model that can support arbitrary dynamics of the short-term rate and the price of risk, it is difficult to envision a sensible economic setup that supports these dynamics. It is easier to understand why the current level of inflation affects risk premia, yet this specification of $\lambda$ (allowing the first element to be a free parameter and setting the other elements to zero) is strongly rejected in the data. This is the test reported in the third row of Table 1's Panel B.

The most parsimonious way to describe the dynamics of risk premia is that the variation in risk premia are proportional to the variation in the one-quarter-ahead expectation of the change in the short rate. From (19) this expectation is

$$E(r_{t+1} - r_t|m_t) = \delta_1'(I - F)c - \delta_1'(I - F)m_t.$$

The restriction is therefore

$$\lambda = \phi\delta_1'(I - F)$$

where the scalar $\phi$ is a free parameter. The final row of Panel B reports that this restriction is not rejected. But this restriction is simply a formal way of saying that short-term and long-term yields tend to move together after a change in inflation.

There is evidence of regime shifts in term structure dynamics.[6] Given this evidence, it is prudent to estimate this model over a shorter sample period that is more homogeneous than is the 1953–2003 period. I therefore estimate the model using data from 1983Q1 through 2003Q4. The results are reported in Table 2.

These results are similar to those for the entire sample period, although details of the dynamics differ. The short rate is more sensitive to inflation in recent data, although it continues to react with a lag. The unrestricted model is not rejected and the restriction $\lambda = 0$ is overwhelmingly rejected. The individual elements of $\lambda$ are not stable across the two samples. A comparison of Fig. 1's Panel B with the corresponding panel in Fig. 3 reveals why. In the 1983–2003 period, the delayed response of long rates occurs in quarter $t + 1$, not quarter $t + 2$. Therefore in this more recent sample the first element of $\lambda$ plays the role that was played by the second element over the longer sample. To pick up the reversal in long-term yields that occurs from $t + 3$ to $t + 4$, the third element of $\lambda$ is approximately the negative of the first element. As in the longer sample, here the sum of all of the elements of $\lambda$ is close to zero. This is consistent with highly transitory effects of inflation on interest rate risk. The main message to take from these additional results is that the dynamics of the price of risk necessary to fit the data are not easy to incorporate into an intuitive model of investor behavior.

# 6 Concluding comments

This paper makes two contributions to the term structure literature. The first contribution is a methodological framework to investigate the relation between the term structure and other, non-yield variables. The framework imposes no-arbitrage without requiring a complete description of the term structure's dynamics. Therefore it can be used to describe the dynamics of expected returns to bonds conditional on the non-yield variables. The framework

[6] Recent research includes Dai, Singleton, and Yang (2003) and Ang and Bekaert (2003).

is simple to implement with GMM.

The second contribution is an investigation of the slow response of both short-term and long-term bond yields to inflation. This slow response implies wide short-run swings in expected returns to long-term bonds. Using the framework introduced in this paper, these swings can be interpreted in the context of a no-arbitrage model. Yet although this pattern can be reconciled with no-arbitrage, the necessary dynamics of the price of interest rate risk are so peculiar that we might be tempted to look for alternative explanations. One alternative is that investors believe that the Fed responds purely contemporaneously to news about inflation. Given these beliefs, delayed reactions consistently surprise investors. But since delayed response is a persistent feature of fifty years of interest rate dynamics, the possibility that investors have such beliefs is also hard to accept.

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Panel A. Parameter estimates

| Parameter | Element of the vector 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\delta_1$ | 0.114 | 0.021 | 0.174 | 0.118 |
| | (0.036) | (0.029) | (0.042) | (0.045) |
| $F_1$ | 0.593 | 0.187 | 0.191 | −0.042 |
| | (0.077) | (0.069) | (0.069) | (0.088) |
| $\lambda$ | −0.250 | 1.156 | −0.270 | −0.589 |
| | (0.192) | (0.256) | (0.346) | (0.228) |

Panel B. Tests of overidentifying restrictions

| Model | LR statistic | d.f. | $p$-val |
|---|---|---|---|
| Unrestricted | 4.351 | 8 | 0.825 |
| $\lambda = 0$ | 16.793 | 4 | 0.002 |
| $\lambda_{[2:4]} = 0$ | 16.791 | 3 | 0.001 |
| $\lambda = \phi\delta_1'(I - F)$ | 3.906 | 3 | 0.272 |

Table 1: Estimates of a no-arbitrage model of inflation and the term structure, 1953-2003

The three-month yield is the sum of a constant and $\delta_1$ times lags zero through three of quarterly inflation. Quarterly inflation follows an AR(4) with parameters $F_1$. Under the equivalent martingale measure the AR(4) of inflation has parameters $F_1 - \lambda$. Estimation is with GMM. The sample period is 1953Q1 through 2004Q4. Standard errors are in parentheses. The test statistics for the restricted models in Panel B are tests relative to the unrestricted model. For the final model listed in Panel B, the restriction implies that the risk adjustment is proportional to the expected quarterly change in the three-month yield.

Panel A. Parameter estimates

| Parameter | Element of the vector 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\delta_1$ | 0.192 | 0.188 | 0.247 | 0.075 |
| | (0.046) | (0.045) | (0.067) | (0.040) |
| $F_1$ | 0.355 | 0.106 | 0.384 | 0.024 |
| | (0.074) | (0.081) | (0.101) | (0.081) |
| $\lambda$ | 0.720 | 0.319 | −0.819 | −0.300 |
| | (0.179) | (0.121) | (0.113) | (0.226) |

Panel B. Tests of overidentifying restrictions

| Model | LR statistic | d.f. | $p$-val |
|---|---|---|---|
| Unrestricted | 5.148 | 8 | 0.742 |
| $\lambda = 0$ | 22.529 | 4 | 0.000 |
| $\lambda_{[2:4]} = 0$ | 15.196 | 3 | 0.002 |
| $\lambda = \phi\delta_1'(I - F)$ | 8.066 | 3 | 0.045 |

Table 2: Estimates of a no-arbitrage model of inflation and the term structure, 1983-2003

The three-month yield is the sum of a constant and $\delta_1$ times lags zero through three of quarterly inflation. Quarterly inflation follows an AR(4) with parameters $F_1$. Under the equivalent martingale measure the AR(4) of inflation has parameters $F_1 - \lambda$. Estimation is with GMM. The sample period is 1983Q1 through 2004Q4. Standard errors are in parentheses. The test statistics for the restricted models in Panel B are tests relative to the unrestricted model. For the final model listed in Panel B, the restriction implies that the risk adjustment is proportional to the expected quarterly change in the three-month yield.

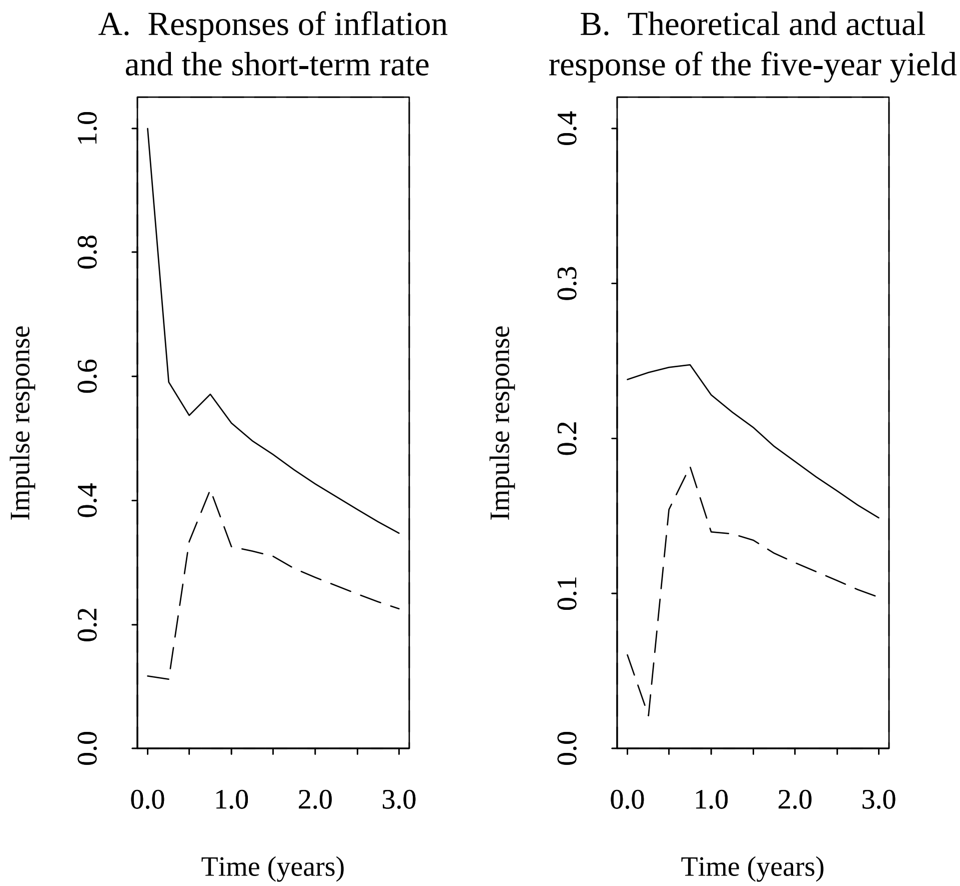


Figure 1: The reaction of inflation and bond yields to inflation

Quarterly inflation is fit to an AR(4) using data from 1953Q1 through 2003Q4. The impulse response of an inflation shock is displayed by the solid line in Panel A. The panel's dashed line is the corresponding impulse response of the three-month Treasury bill yield, based on a regression of the yield on lags zero through three of inflation. This response of the short rate is used to compute the theoretical response of the five-year bond yield under the assumption that risk premia are constant. This theoretical response is the solid line in Panel B. The actual response of the five-year Treasury yield to the inflation shock is the dashed line in Panel B.

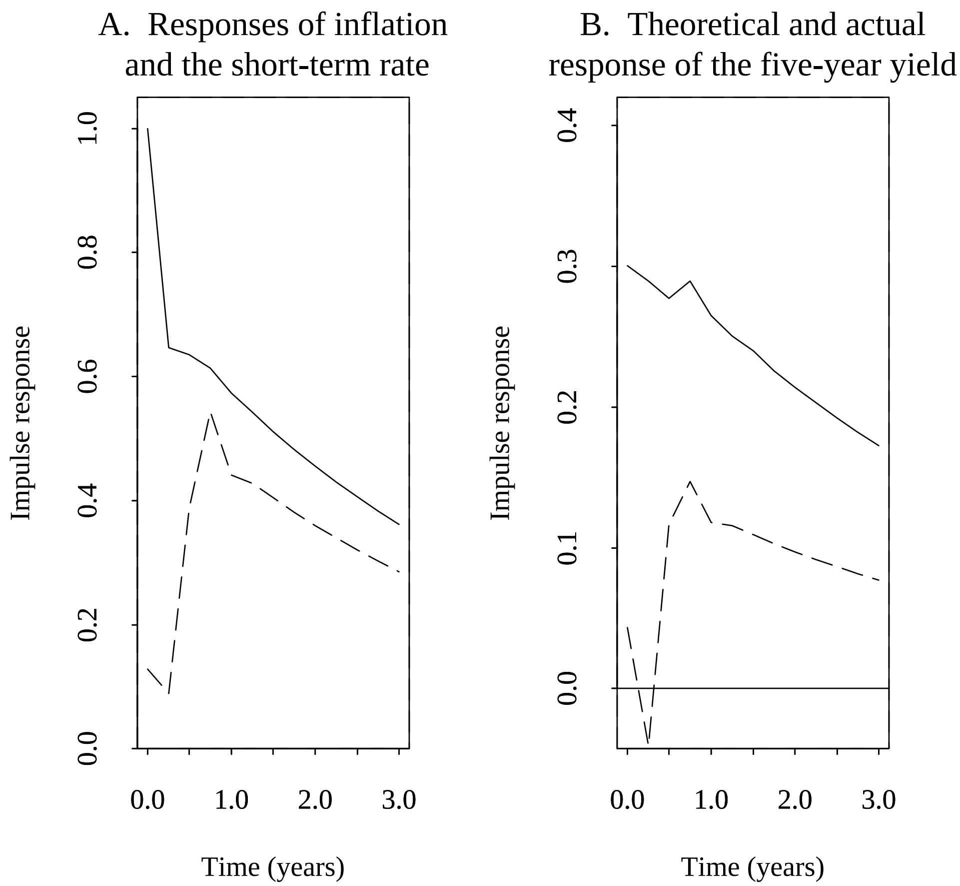


Figure 2: The reaction of inflation and bond yields to inflation: An early sample

This figure duplicates Figure 1 using quarterly data from 1953Q1 through 1982Q4. See the notes to that figure for details.

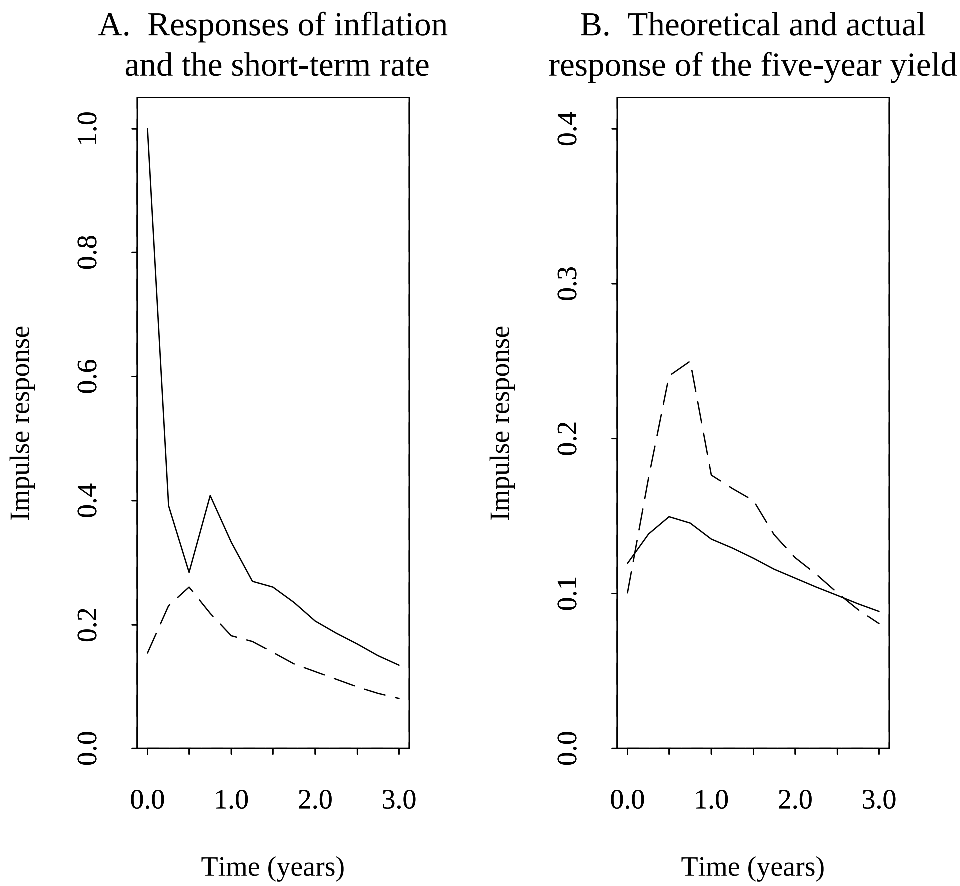


Figure 3: The reaction of inflation and bond yields to inflation: A recent sample

This figure duplicates Figure 1 using quarterly data from 1983Q1 through 2003Q4. See the notes to that figure for details.

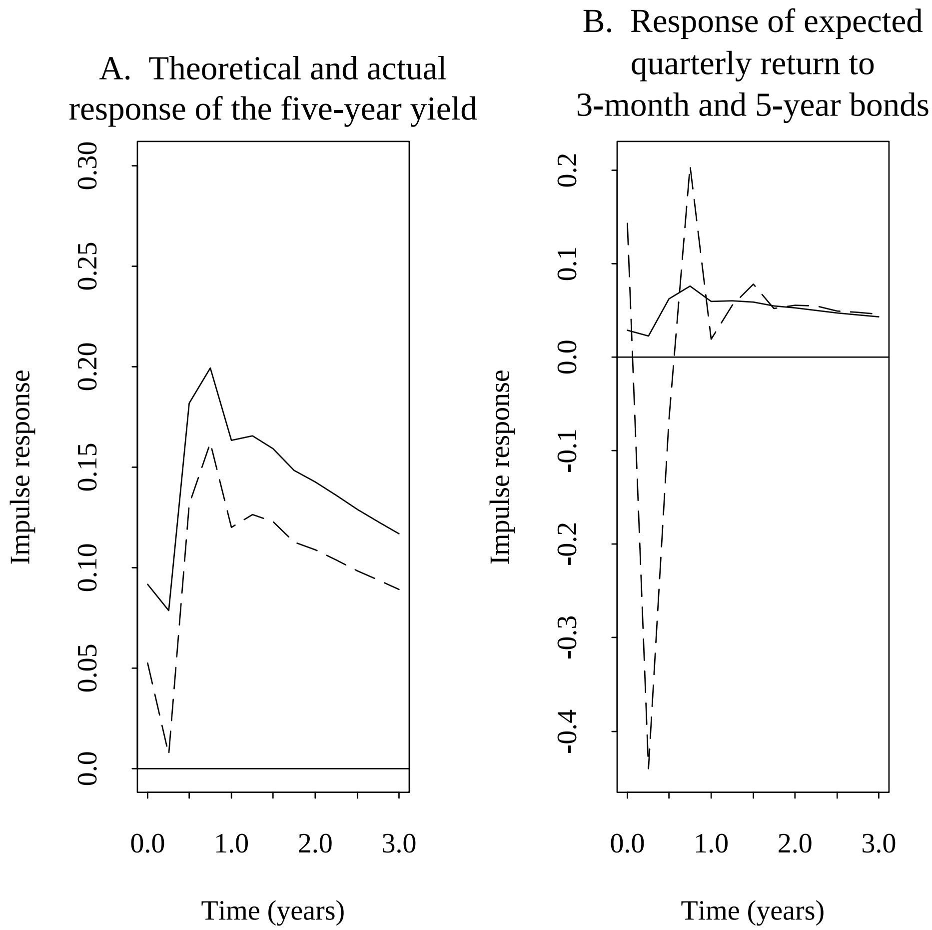


Figure 4: The reaction of bond yields and expected returns to inflation: Results from a no-arbitrage model

Panel A displays the response of the five-year yield to a unit shock to quarterly inflation. The dashed line is the response based on OLS regressions using data from 1953Q1 through 2003Q4 and the solid line is the response implied by the parameters of the no-arbitrage model summarized in Table 1. Panel B displays the response of expected quarterly returns implied by the model's parameters. The solid line is the response of the three-month yield (expressed as a quarterly return) and the dashed line is the response of the expected quarterly return to a five-year bond.