

Investment, Consumption and Hedging under Incomplete Markets*

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Abstract

Entrepreneurs often face undiversifiable idiosyncratic risks from their business investments. Motivated by this observation, we extend the standard real options approach to investment to an incomplete markets environment and analyze the joint decisions of business investments, consumption-saving and portfolio selection. Our analysis depends crucially on whether the investment payoffs are in lump-sum or in flows. Precautionary saving effect plays a key role. In the lump-sum payoff case, risk aversion accelerates investment. Moreover, when the agent's precautionary motive is strong enough, an increase in volatility accelerates investment. These results may be reversed for the flow payoff case. Finally, hedging affects investment decisions by changing the expected growth of wealth and reducing the agent's exposure to idiosyncratic risk. The agent's hedging demand is higher when he is closer to exercising the investment option.

Keywords: real options, idiosyncratic risk, hedging, risk aversion, precautionary saving, incomplete markets

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1 Introduction

A real investment often has three important characteristics. First, it is often partially or completely irreversible. Second, its future rewards are uncertain. Finally, the investment time is to some extent flexible. In the last three decades, a voluminous literature has developed that aims to study the implications of these three characteristics for the investment decision.¹ A key insight of this literature is to view making an investment decision as exercising an American style call option, where “American style” refers to the flexibility of choosing the time of option exercise. Based on this analogy and the seminal contribution on option pricing by Black and Scholes (1973) and Merton (1973), one can apply financial option theory to analyze the irreversible investment decision. This real options approach to investment has become a workhorse in modern economics and finance.

This real options approach relies on one of the following assumptions: *(i)* the real investment opportunity is tradable; *(ii)* its payoff can be spanned by existing traded assets; or *(iii)* the agent is risk neutral. However, these assumptions are violated in many applications. For example, consider entrepreneurial activities. Entrepreneurs combine their business investment opportunities and ideas with their skills to generate economic profits. While entrepreneurs may have valuable projects, these projects may not be freely traded or their payoffs may not be spanned by existing assets because of liquidity restrictions or the lack of liquid markets. These capital market imperfections may be due to moral hazard, adverse selection, transactions costs, or contractual restrictions.² Thus, the investment opportunities may have substantial undiversifiable idiosyncratic risks. Owning them exposes entrepreneurs to these undiversifiable risks. Consequently, entrepreneurs’ well-being depends heavily on the outcome of their investments. Moreover, entrepreneurs’ attitudes towards risk should play an important role in determining their consumption-saving, portfolio selection, and investment decisions.³

Motivated by examples such as entrepreneurial activities, we extend the standard real options approach to investment to allow for incomplete markets. We use a utility maximization framework in which the agent chooses consumption and portfolio rules, as well as undertakes an irreversible investment to obtain income. We start with a simple case where the agent can only trade a risk-free asset to insure himself against the project risk. We then consider the case

¹Arrow (1968) and Bernanke (1983) are among early contributions on irreversible investment. For early stochastic continuous-time models, see Brennan and Schwartz (1985), McDonald and Siegel (1986), and Pindyck (1988). Abel and Eberly (1994) provide a unified model of (incremental) investment under uncertainty. Dixit and Pindyck (1994) provide a textbook treatment of important contributions to this literature.

²Grenadier and Wang (2005) analyze a real options model with agency issues.

³There is a fast growing literature on empirical evidence for entrepreneurship. See Gentry and Hubbard (2004), Heaton and Lucas (2000), and Moskowitz and Vissing-Jorgensen (2002), among others.

where the agent can also trade a risky asset to partially hedge against the project risk.

To facilitate the discussions of our results, we use real estate development as a concrete example. The land value may be viewed as the option value of developing real estate.⁴ No property can be built without land and land is of limited supply. Thus, land captures much of the value added in the property, particularly in regions with a scarce supply of land. Real estate is a suitable application of our model, because (i) land is primarily held by noninstitutional investors such as individuals and private partnerships (Williams (2001)) and because (ii) individuals and private partnerships are subject to undiversifiable idiosyncratic risks more than institutional investors like pension fund firms and life insurance companies.

To further illustrate our model, consider a real estate entrepreneur's development decisions. While he owns the land and will choose the time to build the property, he may either sell the property or continue to manage the property after developing it.⁵ When he pays the construction cost and sells the property upon the completion of development, he then receives a lump-sum sale price. We dub this situation the lump-sum payoff case. Alternatively, the real estate entrepreneur may not only be the developer, but also the manager. The entrepreneur may be the most qualified manager, because he can locate the tenants with the highest willingness to pay and because he can maintain the property at the lowest operating costs. Therefore, it may still make economic sense for the developer to manage the property after construction is complete, even though he will face additional undiversifiable idiosyncratic property risks after development. Under this setting, the developer receives a perpetual stream of uninsurable rental payoffs (in excess of operating costs) from managing the property after development. We dub this scenario the flow payoff case.

Under complete markets, the entrepreneur can fully diversify the idiosyncratic property risks. One can take the risk-adjusted present discounted value of future cash flows as the market sale value, and thus there is no distinction between the preceding two scenarios. However, when the investment opportunity is not tradable and not spanned by existing traded assets, the standard replicating and no arbitrage argument does not apply. We thus follow the certainty equivalent approach in the literature on the pricing of nontraded assets to value cash flows by analyzing the entrepreneur's utility maximization problem.⁶ We show that the lump-sum

⁴See Titman (1985), Williams (1991), and Grenadier (1996) for applications of the real options approach to real estate development.

⁵Of course, choosing whether to sell or manage the property is in itself a decision. We assume this decision is exogenous in the paper.

⁶See Carpenter (1998), Detemple and Sundaresan (1999), Hall and Murphy (2000), Kahl, Liu and Longstaff (2003), among others, on nontraded asset valuation such as employee stock options. See Section 2 for further discussions.

and flow payoff cases deliver different economic predictions, and hence the equivalence between these two cases no longer holds.

We start with the lump-sum payoff case. We first analyze the effect of risk attitude. To derive analytically tractable solutions, we adopt the constant absolute risk aversion (CARA) utility specification. For this utility, the risk aversion parameter also measures the precautionary saving motive (Kimball (1990)). We show that a stronger precautionary saving motive implies lower certainty equivalent wealth associated with the investment opportunity. We may also interpret this certainty equivalent wealth as the implied option value. Thus, risk aversion lowers the implied option value and speeds up investment.

We next analyze the impact of risk. Unlike the standard real options analysis, an increase in the payoff volatility has two effects on the development decision. One is the standard option effect in that it raises the option value of waiting. The other is the precautionary saving effect in that a more volatile payoff raises precautionary savings demand. The precautionary saving effect implies that a higher idiosyncratic volatility lowers the certainty equivalent wealth associated with the investment option or the implied option value, *ceteris paribus*. Therefore, the net effect of volatility on investment is ambiguous, depending on which one of the preceding effects dominates. In particular, if the precautionary motive is sufficiently strong, then the entrepreneur may build the property earlier when volatility is higher. This result is to the opposite of the prediction in standard real options models.

Finally, we analyze the role of hedging. Hedging reduces the agent's exposure to the project's idiosyncratic risk and thus reduces the precautionary saving effect on the implied option value, which in turn makes the investment option more valuable. In addition, hedging also changes the risk-adjusted growth of the payoff process. This risk-adjusted growth effect raises the implied option value when the correlation between the market risk and the project risk is negative, since the entrepreneur holds a long position in the market portfolio. By contrast, this effect is negative when the correlation is positive, since the entrepreneur holds a short position in the market portfolio. When the entrepreneur can trade an additional risky asset to span the idiosyncratic risk and hence to eliminate the negative precautionary saving effect, the implied option value increases. Therefore, the entrepreneur invests later under complete markets than under incomplete markets.

We now turn to the flow payoff case. In our previous real estate development example, this corresponds to the case where the developer also manages the real estate after its completion. We show that the previous results may be reversed in the flow payoff case. This is because the developer still faces undiversifiable idiosyncratic risk from the payoff stream after exercising

the investment option, and hence he values this payoff stream as certainty equivalent wealth. Thus, the previously discussed precautionary saving effect and the risk-adjusted growth effect also influence the implied project value after exercising the investment option. These effects may dominate those on the implied option value, and hence cause the reversal of the results in the lump-sum payoff case.

In addition to contributing to the investment (real options) literature, our paper also contributes to the portfolio choice literature. Building on the insights behind the Black-Merton-Scholes analysis, we study hedging against endogenously timed income under incomplete markets.⁷ We show that the hedging demand increases with the investment option delta.⁸ Since the option delta increases in the underlying project payoff value, our model predicts that the developer's hedging demand increases when his development option gets closer to being "in the money." With regard to the consumption-saving literature, we extend the standard incomplete markets analysis to allow the agent to endogenously determine the timing of his income process. We show that volatility not only has a negative effect on consumption, but also a positive option effect due to the endogeneity of the income timing choice. Our paper also relates to Henderson (2005) and Hugonnier and Morellec (2005). They assume that the agent maximizes expected wealth at the time of investment. While both papers study real options models under incomplete markets, neither paper studies the agent's consumption decision and its interaction with investment and portfolio choice decisions.

The remainder of the paper proceeds as follows. Section 2 analyzes a self insurance model when the payoff from real investment is given in lump sum. Section 3 generalizes the model in Section 2 to allow for the hedging opportunity. Section 4 extends the models in Sections 2 and 3 to settings in which the real investment payoffs are given in flows. Section 5 concludes. Technical details are relegated to appendices.

2 A Self-Insurance Model with Lump-sum Payoff

This section provides a simple model that allows us to develop intuition on how the agent's attitude towards risk affects his investment decisions when markets are incomplete. In order to achieve this objective in the simplest possible setting, we integrate a canonical consump-

⁷See Svensson and Werner (1993), Duffie et al. (1997), Koo (1998), Viceira (2001), Heaton and Lucas (2000), Davis and Willen (2002), and Chapter 6 in Campbell and Viceira (2002), among others, on dynamic consumption and portfolio choice when an investor is endowed with nontraded stochastic income.

⁸Delta is defined as the change in the investment option value for a unit increase of the underlying project payoff value.

tion/saving model with a standard real options based irreversible investment model.⁹

2.1 Model Setup

Time is continuous and the horizon is infinite. There is a single perishable consumption good (the numeraire). The agent derives utility from a consumption process C according to

$$E \left[\int_0^\infty e^{-\beta t} U(C_t) dt \right], \quad (1)$$

where U is an increasing and concave function and $\beta > 0$ is his discount rate. For expositional convenience, we assume that β is equal to r , the risk-free interest rate.¹⁰

The agent owns an investment project and can undertake this project irreversibly at some endogenously chosen time τ . Note that the investment time τ is stochastic from today's perspective. The investment costs $I > 0$. The agent pays this cost only at the investment time τ . This cost is financed from the agent's own wealth. If there is a shortage of fund, the agent may borrow at the risk-free rate r . In order to focus on the effect of market incompleteness in the simplest possible setting, we do not consider borrowing constraints or costly external financing. Instead, we impose the conventional transversality condition for the agent to rule out Ponzi games. After the agent exercises the investment option at time τ , the project generates a lump-sum payoff X_τ . We also assume that the payoff process X is governed by an arithmetic Brownian motion process

$$dX_t = \alpha_x dt + \sigma_x dZ_t, \quad X_0 \text{ given}, \quad (2)$$

where α_x and σ_x are positive constants and Z is a standard Brownian motion.¹¹ This process implies that payoffs may take negative values. We interpret negative values as losses.

As discussed earlier, investing in the project is analogous to exercising an American-style option, in the sense that the agent has the right but not the obligation to invest at some future time of his choosing. However, unlike financial options, the real option here cannot be traded in the market, and hence the agent is subject to undiversifiable project risk. He can trade a risk-free asset only to insure himself against this risk. Let $\{W_t : t \geq 0\}$ denote a wealth process.

⁹See Leland (1968) for early studies on precautionary savings. See Zeldes (1989), Caballero (1991b), and Deaton (1991) for dynamic incomplete markets consumption models. See Brennan and Schwartz (1985), McDonald and Siegel (1986), and Dixit and Pindyck (1994) for standard real options models.

¹⁰It is straightforward to extend our analysis to allow for differences between the agent's subjective discount rate and the interest rate. We choose not to, however, because no additional insight will be gained for the issue that we are after.

¹¹Unlike the often adopted geometric Brownian motion process, the specification in (2) proves more convenient within our setup. Wang (2005) derives a closed-form consumption-saving rule using affine processes and exponential utility.

Then the wealth dynamics are given by

$$dW_t = (rW_t - C_t) dt, \quad W_0 \text{ given.} \quad (3)$$

That is, the agent accumulates wealth at the rate of $(rW_t - C_t)$, the difference between the interest income rW_t and consumption rate C_t . At the investment time τ , the agent pays the investment cost I and obtains the lump-sum payoff X_τ , and hence his wealth is raised by the amount $(X_\tau - I)$. That is, the agent's wealth jumps by a discrete amount $(X_\tau - I)$ at τ , in that $W_\tau = W_{\tau-} + X_\tau - I$, where $W_{\tau-}$ and W_τ denote the agent's wealth just before and immediately after the agent exercises the investment option, respectively. The agent's optimization problem is to choose both his investment timing strategy τ and consumption process C to maximize his utility given in (1) subject to (3) and to a transversality condition specified later.

2.2 Model Solution

We solve the agent's problem by working backwards using dynamic programming. We first consider the problem after the agent exercises the investment option. In this case, the agent's optimization problem is a standard deterministic consumption-saving problem without labor income. Let $V^0(w)$ be the corresponding value function. By a standard argument, $V^0(w)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:¹²

$$rV^0(w) = \max_{c \in \mathbb{R}} U(c) + (rw - c)V_w^0(w). \quad (4)$$

Under the deterministic setting, the agent's consumption is constant over time and is equal to the annuity value rw of his wealth, and therefore, his wealth remains constant at w at all times. This is the familiar consumption smoothing result.¹³

We next consider the case before the option is exercised. It is worth noting that the agent's value function depends on both his wealth w and the current value x of his investment opportunity. Let $V(w, x)$ denote the corresponding value function. The standard dynamic programming argument implies that $V(w, x)$ satisfies the following HJB equation:

$$rV(w, x) = \max_{c \in \mathbb{R}} U(c) + (rw - c)V_w(w, x) + \alpha_x V_x(w, x) + \frac{\sigma_x^2}{2} V_{xx}(w, x). \quad (5)$$

The above HJB equation is similar to an asset pricing equation. It states that the agent chooses his consumption optimally by setting the return $rV(w, x)$ of his value function to equal the sum

¹²The transversality condition $\lim_{T \rightarrow \infty} [e^{-rT} J(W_T)] = 0$ must also be satisfied.

¹³This result follows from two steps: (i) the equality between the agent's discount rate and the interest rate implies that the marginal utility is constant at all times ($U'(C_t) = U'(C_s)$); (ii) The strict concavity of the utility function further implies that $C_t = C_s$.

of his instantaneous utility $U(c)$ and the total expected changes of his value function (due to the change in wealth and also in the investment opportunity).

We now specify boundary conditions. First, the no-bubble condition $\lim_{x \rightarrow -\infty} V(w, x) = V^0(w)$ must be satisfied. This condition states that when the investment payoff goes to negative infinity, the agent will never exercise the investment option and his value function is equal to that without the investment option. Next, as is standard in the optimal stopping problems, at the instant of investment, the following value-matching condition must hold:

$$V(w, x) = V^0(w + x - I). \quad (6)$$

This equation implicitly defines an investment boundary $x = \bar{x}(w)$. In general, this boundary $\bar{x}(w)$ depends on the agent's wealth level w . Finally, because this boundary is chosen optimally, the following smooth-pasting condition must be satisfied:¹⁴

$$\left. \frac{\partial V(w, x)}{\partial x} \right|_{x=\bar{x}(w)} = \left. \frac{\partial V^0(w + x - I)}{\partial x} \right|_{x=\bar{x}(w)}, \quad (7)$$

$$\left. \frac{\partial V(w, x)}{\partial w} \right|_{x=\bar{x}(w)} = \left. \frac{\partial V^0(w + x - I)}{\partial w} \right|_{x=\bar{x}(w)}. \quad (8)$$

The first smooth-pasting condition (7) states that the marginal change of the investment opportunity has the same marginal effect on the agent's value functions just before and immediately after exercising the option. Similarly, the second smooth-pasting condition (8) states that the marginal effect of wealth must be the same on the agent's value functions just before and immediately after exercising the option. Unlike the conventional irreversible investment models (Dixit and Pindyck (1994)), here the agent's wealth enters as an additional state variable, which gives rise to the second smooth-pasting condition (8).

Technically speaking, the agent's optimization problem is a combined control and stopping problem, which is generally difficult to solve. To simplify the problem, we assume that the agent's preferences are represented by a constant absolute risk aversion (CARA) utility $U(c) = -e^{-\gamma c}/\gamma$. The parameter $\gamma > 0$ is the coefficient of absolute risk aversion. It is also equal to the coefficient of absolute prudence $-U'''(c)/U''(c)$, which captures the precautionary saving motive. It is well known that consumption and precautionary savings under incomplete markets depend crucially on the convexity of marginal utility. Since our objective is to analyze the interaction between consumption-saving and investment decisions under incomplete markets, we naturally choose CARA utility as a starting point.

¹⁴See, for example, Krylov (1980), Dumas (1991) and Dixit and Pindyck (1994).

Importantly, CARA utility has no wealth effect and permits an analytically tractable solution to the value functions and policy functions. It also implies that the investment boundary is flat, in that $\bar{x}(w)$ is independent of wealth w . The following proposition summarizes the solution to the agent's combined consumption (control) and investment (stopping) problem.

Proposition 1 *The agent exercises the investment option the first time the process X hits the threshold \bar{x} from below. After exercising the option, the agent's consumption rule is given by $\bar{c}(w) = rw$. Before exercising the option, his consumption rule is given by*

$$\bar{c}(w, x) = r(w + G(x)), \quad (9)$$

where $(G(x), \bar{x})$ is the solution to the following free boundary problem:

$$rG(x) = \alpha_x G'(x) + \frac{\sigma_x^2}{2} G''(x) - \frac{\gamma r \sigma_x^2}{2} G'(x)^2, \quad (10)$$

subject to the no-bubble condition $\lim_{x \rightarrow -\infty} G(x) = 0$, and the boundary conditions:

$$G(\bar{x}) = \bar{x} - I, \quad (11)$$

$$G'(\bar{x}) = 1. \quad (12)$$

Moreover, G is increasing and convex.

In the next subsection, we analyze the intuition behind this proposition and discuss its implications.

2.3 Effects of Risk and Risk Attitude

Consumption. We first consider the agent's consumption policies. After exercising the option, the agent solves a deterministic consumption smoothing problem. As noted earlier, his wealth remains constant and consumption is equal to the interest income at all times. Before exercising the option, the agent's consumption rule (9) is given by the annuity value of the sum of his financial wealth w and a term $G(x)$ related to investment payoffs. One can interpret $G(x)$ as the certainty equivalent wealth derived from the agent's investment opportunity. Specifically, we follow the consumption literature to define certainty equivalent wealth as the value w_{ce} satisfying the equation $V^0(w + w_{ce}) = V(w, x)$; that is, the agent is indifferent between the situation where he receives stochastic income in the future and the situation where he has no income but a total wealth level of $(w + w_{ce})$. Given the functional forms of V^0 and V derived in the appendix, we have $w_{ce} = G(x)$.

Note that even though the agent does not receive payoff x before exercising the option, he rationally anticipates his investment opportunity. Thus, the future investment payoff matters for his current consumption as well as for his lifetime well-being. This idea is at the core of the permanent-income/precautionary-saving literature. As in this literature, the certainty equivalent wealth $G(x)$ captures the agent's precautionary savings demand. This demand is reflected by the nonlinear term on the right side of equation (10). As evidenced by the negative sign of this nonlinear term, precautionary savings lower $G(x)$. Our model thus extends the forward-looking consumption smoothing intuition to an incomplete markets setting with stochastic income derived from investment decisions. Given this, one should expect that investment and consumption decisions are interdependent. We next turn to this question.

Implied Option Value and Investment Decision. Proposition 1 demonstrates that the certainty equivalent wealth $G(x)$ solves a free-boundary problem (10)-(12). These equations are similar to the valuation equations and boundary conditions in the standard real option models of McDonald and Siegel (1986) and Dixit and Pindyck (1994). Based on this similarity, we interpret x as the project value and the certainty equivalent wealth $G(x)$ as the implied option value to invest in the underlying project. Although under incomplete markets there is no unique stochastic discount factor to define a market option value for nontraded investment opportunities, our interpretation can be justified by adopting the certainty equivalent valuation methodology, which is widely used in the literature on the pricing of nontraded assets. Specifically, we follow this literature by defining the implied option value Q of the project as the solution to the equation $V(w - Q, x) = V^0(w)$; that is, the agent is indifferent between the situation where he has no investment opportunity and the situation where he pays the price Q and obtains the investment opportunity. Given the functional form of V and V^0 in the appendix, one can show that $Q = G(x)$.

We emphasize that the two interpretations of $G(x)$ – the certainty equivalent wealth and the implied option value – are the same in our setup. This is due to the absence of a wealth effect under CARA utility. We will thus use certainty equivalent wealth (from the consumption literature perspective) and implied option value interchangeably throughout the remainder of the paper.

Before discussing how preferences influence the implied option value $G(x)$ under incomplete markets, we first briefly recapitulate the solution of a standard real options model under risk neutrality. We obtain this solution by setting $\gamma = 0$ in equation (10),

$$G(x) = \frac{1}{\lambda_0} e^{\lambda_0(x-\bar{x})}, \quad \text{for } x \leq \bar{x}, \text{ and } \bar{x} = I + \frac{1}{\lambda_0}, \quad (13)$$

where $\lambda_0 = -\sigma_x^{-2}\alpha_x + \sqrt{\sigma_x^{-4}\alpha_x^2 + 2r\sigma_x^{-2}}$ for $\sigma_x > 0$, and $\lambda_0 = r/\alpha$ for $\sigma_x = 0$. As is well known, the agent's investment decision is characterized by a trigger policy whereby the agent invests the first time the investment payoff is larger than the threshold level \bar{x} . While our model allows for precautionary saving motive to affect the investment decision, the investment policy in our model is also characterized by the threshold policy. Therefore, throughout the paper, any statement with regard to the investment decision is about the investment threshold \bar{x} .¹⁵

The main difference between our model and the preceding standard real options model under risk neutrality is that the implied option value $G(x)$ depends not only on the parameters describing the asset value such as the risk-free rate r , drift α_x and volatility σ_x , but it also depends on the agent's risk aversion or precautionary motive measured by γ . The latter dependence captures the notion that the agent's risk attitude matters not only for consumption decisions, but also for investment decisions when markets are incomplete. This is reflected by the precautionary savings demand, via the nonlinear term on the right side of (10). Since this term appears with a negative sign, the implied option value $G(x)$ is lower when the precautionary motive is stronger, *ceteris paribus*. Since the project payoff value x does not depend on the agent's risk attitude, the net effect of an increase in γ is to encourage earlier investment. Figure 1 plots the implied option value $G(x)$ versus the value of the underlying investment opportunity x for two values of γ . This figure illustrates that the investment threshold decreases with the agent's precautionary motive or risk aversion γ .

[Insert Figure 1 Here]

Approximate Solution. To gain further intuition, we use the asymptotic approximation method to compute approximate solutions for the implied option value $G(x)$ and the investment threshold \bar{x} .¹⁶ We expand the option value $G(x)$ and the investment threshold \bar{x} to the first order of σ_x^2 , in that $G(x) \approx G_0(x) + G_1(x)\sigma_x^2$ and $\bar{x} \approx \bar{x}_0 + \delta_1\sigma_x^2 \equiv \bar{x}_1$. Plugging these expansions in (10)-(12), we show in the appendix that $\bar{x}_0 = I + \alpha_x/r$ and

$$\bar{x}_1 = \bar{x}_0 + \left(\frac{1}{\alpha_x} - \gamma\right) \frac{\sigma_x^2}{2}. \quad (14)$$

This approximate solution indicates that, to a first-order approximation with respect to σ_x^2 , a stronger precautionary motive (higher γ) lowers the investment threshold, consistent with our earlier discussions based on the non-linear ODE (10) and the boundary conditions (11)-(12).

¹⁵A larger value of threshold implies delayed investment since the average waiting time to invest for the process $(X_t)_{t \geq 0}$ starting from a point $X_0 < \bar{x}$ is given by $(\bar{x} - X_0)/\alpha_x$. Alternatively, one can also interpret that a larger value of threshold implies less investment as in Dixit and Pindyck (1994).

¹⁶See Judd (1998). Kogan (2001) applies this method to solve an irreversible (incremental) investment model.

The above approximate solution also helps us to understand the effect of volatility. An increase in volatility σ_x has two opposing effects. First, it raises the implied option value $G(x)$. This is the result of the standard option effect, which comes from the convex payoff feature of options. The agent’s investment opportunity resembles an American style call option. He can capture the upside gains and limit the downside losses by simply waiting until the option is sufficiently “in the money.” Second, an increase in σ_x raises the precautionary savings demand and hence lowers the implied option value $G(x)$. Both effects are reflected in the last term on the right side of (14). When γ is sufficiently small, the option effect dominates the precautionary saving effect. Thus, an increase in volatility σ_x raises the implied option value and delays investment, as predicted by the standard real options approach. Surprisingly, it follows from (9) that consumption before the option exercise also increases with volatility. This effect is not present in the consumption literature. By contrast, when γ is sufficiently large, the precautionary saving effect may dominate the option effect. Consequently, the preceding results are reversed. In summary, our analysis of the impact of volatility on consumption and investment suggests that the prediction of the standard real options approach may be misleading when applied to entrepreneurial investments, since entrepreneurs often face large undiversifiable idiosyncratic risks.

Finally, we use numerical solutions to conduct further analysis. We apply the projection method detailed in the appendix to solve the free boundary problem characterized by (10)-(12). We find that, for a small σ_x , our preceding approximate solution is very close to the “true” solution delivered by the projection method. For a large range of parameter values, Figure 2 plots the investment threshold as a function of the volatility σ_x and the parameter γ . This figure demonstrates that our preceding results and intuition also extend to general parameter values.

[Insert Figure 2 Here]

3 Lump-sum Payoff Case with Hedging Opportunities

In the previous section, the agent can trade only a risk-free asset to insure himself against the project risk. We now allow the agent to trade a risky asset to partially hedge against the project risk. This asset represents a market portfolio. We focus on the role of this hedging opportunity in investment timing.

3.1 The Model

Let $\{P_t : t \geq 0\}$ denote the risky asset's price process and assume that the return is governed by the following process:

$$dP_t/P_t = \mu_e dt + \sigma_e dB_t, \quad (15)$$

where μ_e and σ_e are positive constants, and B is a standard Brownian motion correlated with the Brownian motion Z , which drives the innovations of the project payoff as given in (2). Let $\rho \in (-1, 1)$ be the correlation coefficient between the return on the risky asset and the agent's project payoff, and let $\eta = (\mu_e - r)/\sigma_e > 0$ denote the Sharpe ratio of the market portfolio.

One can alternatively rewrite the observed payoff process $\{X_t : t \geq 0\}$ given in (2) as follows:

$$dX_t = \alpha_x dt + \rho \sigma_x dB_t + \sqrt{1 - \rho^2} \sigma_x d\tilde{B}_t, \quad (16)$$

where B and \tilde{B} are two independent standard Brownian motions. One may think of B as the Brownian motion describing the systematic (market) risk, and thus $\rho \sigma_x$ is the systematic component of the volatility for the project payoff. One may then interpret \tilde{B} as the Brownian motion describing the idiosyncratic project risk, and thus $\sqrt{1 - \rho^2} \sigma_x$ is the idiosyncratic component of the volatility for the project payoff. Hedging reduces the agent's exposure to idiosyncratic risk. The absolute value of the correlation coefficient $|\rho|$ measures the extent of hedging opportunities.

Let π_t be the amount allocated to the risky asset at time t , measured in units of the consumption good. The agent's problem is to choose a consumption process C , a portfolio allocation rule π , and an investment timing strategy τ to maximize his utility (1) subject to his wealth dynamics:

$$dW_t = (rW_t + \pi_t(\mu_e - r) - C_t) dt + \pi_t \sigma_e dB_t, \quad W_0 \text{ given.} \quad (17)$$

Similar to Section 2, the agent's wealth jumps immediately after he invests, in that $W_\tau = W_{\tau-} + X_\tau - I$, where $W_{\tau-}$ and W_τ are his wealth just before and immediately after his investment at time τ , respectively. Note that (17) is the same both before and after the option exercise.

We use the same dynamic programming method as in Section 2 to solve the agent's problem and present the solution below.

Proposition 2 *The agent exercises the investment option the first time the process X hits the threshold \bar{x} from below. After exercising the option, the optimal consumption and portfolio rules*

are given by

$$\bar{c}(w) = r \left(w + \frac{\eta^2}{2\gamma r^2} \right), \quad (18)$$

$$\bar{\pi}(w) = \frac{\eta}{\gamma \sigma_e} \frac{1}{r}. \quad (19)$$

Before exercising the option, the optimal consumption and portfolio rules are given by

$$\bar{c}(w, x) = r \left(w + G(x) + \frac{\eta^2}{2\gamma r^2} \right), \quad (20)$$

$$\bar{\pi}(w, x) = \frac{\eta}{\gamma \sigma_e} \frac{1}{r} - \frac{\sigma_x \rho}{\sigma_e} G'(x), \quad (21)$$

where (G, \bar{x}) is the solution to the following free boundary problem:

$$rG(x) = (\alpha_x - \rho\eta\sigma_x) G'(x) + \frac{\sigma_x^2}{2} G''(x) - \frac{\gamma r \sigma_x^2 (1 - \rho^2)}{2} G'(x)^2, \quad (22)$$

subject to the no-bubble condition $\lim_{x \rightarrow -\infty} G(x) = 0$, and also the boundary conditions

$$G(\bar{x}) = \bar{x} - I, \quad (23)$$

$$G'(\bar{x}) = 1. \quad (24)$$

Moreover, G is increasing and convex.

Note that, when $\rho = 0$, the solution for (G, \bar{x}) reduces to that given in Proposition 1. This is intuitive because when the project risk is not correlated with the market, the risky asset does not provide any insurance to the agent. The agent effectively insures himself against the investment risk and hence makes the same investment decision as in Section 2. In the next subsection, we analyze the intuition behind Proposition 2 and focus on the role of hedging.

3.2 Implications of Hedging

The consumption rule (18) and the portfolio rule (19) after the option exercise are solutions to the standard Merton style consumption-portfolio choice problem with CARA utility (Merton (1969)). Equation (19) gives the standard mean-variance efficient rule for CARA utility. After exercising the option, the agent has no hedging demand since the lump-sum project payoff has been completely realized at the exercising time τ .

Consider next the agent's consumption and portfolio rules (20)-(21) before the option exercise. Similar to the self-insurance model with lump-sum payoffs in Section 2, we may interpret $G(x)$ as the certainty equivalent wealth associated with the investment option. Alternatively,

we can interpret $G(x)$ as the implied option value. Since the risky asset is correlated with the project payoff, the agent hedges against the stochastic evolution of the project payoff X . The hedging demand is given by the second term in the asset allocation rule (21). Intuitively, the certainty equivalent wealth $G(x)$ is higher, when the project payoff x is higher. This intuition suggests $G'(x) > 0$. Equation (21) then implies that the agent has a greater demand for hedging, when the change of investment opportunities has a larger impact on consumption via the certainty equivalent wealth $G(x)$.

We may interpret $G'(x)$ as the option Δ for the underlying project, by drawing the analogy to the complete-markets Black-Merton-Scholes analysis. It is worth emphasizing that the implied option value $G(x)$ is not only increasing in project payoff x , but also is convex in project payoff x . The convexity of the implied option value $G(x)$ implies that option Δ is increasing in the underlying project payoff x . This result implies that the agent's hedging demand $|\rho G'(x)\sigma_x/\sigma_e|$, the second term in the portfolio rule (21), increases with x . Therefore, his hedging demand is higher when his investment option is closer to being "in the money," in that x is closer to the investment threshold \bar{x} . It is worth emphasizing that, to the best of our knowledge, our results provide the first option Δ analysis under an incomplete markets real options setting.

The differential equation (22) and boundary conditions (23)-(24) indicate that the hedging opportunity has two important effects on the implied option value $G(x)$ and the investment threshold \bar{x} : the precautionary saving effect and the risk-adjusted growth effect. The precautionary saving effect captures the conventional wisdom that hedging allows for diversification, which reduces the agent's exposure to idiosyncratic risk. In our model, this effect is reflected in the last precautionary saving term in (22). Compared to the model without hedging opportunity in Section 2, here hedging reduces the agent's exposure to idiosyncratic risk from σ_x to $\sqrt{1 - \rho^2} \sigma_x$. Therefore, hedging reduces the precautionary savings, because the precautionary saving effect depends on the idiosyncratic volatility $\sqrt{1 - \rho^2} \sigma_x$.

In addition to the precautionary saving effect, hedging also has a risk-adjusted growth effect on the implied option value $G(x)$, as reflected by the first term on the right side of the differential equation (22). Specifically, because there is a hedging demand before the option exercise, hedging affects the drift of the payoff process. That is, for the purpose of computing $G(x)$, it is the drift $(\alpha_x - \rho\eta\sigma_x)$, rather than α_x that matters for the implied option value $G(x)$. When $\rho > 0$, hedging lowers the drift and hence reduces the implied option value. The intuition is the following. When the project risk and the market risk move in the same direction, the agent hedges against the project risk by holding a short position in the risky asset, as shown by the second term in the portfolio rule (21). A higher correlation ρ implies that the agent holds a

larger short position, *ceteris paribus*. Since a short position lowers the agent's wealth, a larger extent of hedging opportunity (i.e., a larger ρ) is more costly in terms of wealth reduction.

The above analysis implies that the precautionary saving effect and the risk-adjusted growth effect work oppositely on the implied option value $G(x)$, when $\rho > 0$. Therefore, the net impact of these two opposing effects on the implied option value $G(x)$ and the investment threshold \bar{x} is ambiguous. When the agent's precautionary motive is strong enough with a sufficiently large γ , the precautionary saving effect on $G(x)$ may dominate the risk-adjusted growth effect. Consequently, a larger extent of hedging opportunity may raise the implied option value $G(x)$ and delay the option exercise. We illustrate this result in Figure 3a for $\gamma = 150$. This figure indicates that the implied option value with hedging opportunities ($\rho = 0.9$) is higher than without hedging opportunities ($\rho = 0$), when the payoff x is close to the threshold values. By contrast, Figure 3b illustrates that for $\gamma = 1$, the implied option value with hedging opportunities ($\rho = 0.9$) is lower than without hedging opportunities ($\rho = 0$).

[Insert Figure 3 Here]

While we have discussed extensively the case of $\rho > 0$, we now turn to the other case of $\rho < 0$. As for the case of $\rho > 0$, an increase in hedging opportunity (measured by a larger $|\rho|$) reduces idiosyncratic volatility and raises the implied option value $G(x)$. However, the risk-adjusted growth effect of hedging is the opposite to that in the case of $\rho > 0$. The intuition is as follows. When $\rho < 0$, the agent hedges against the project risk by holding a long position in the risky asset. Since the long position increases the agent's wealth in expectation, hedging raises the certainty equivalent wealth $G(x)$. Therefore, for $\rho < 0$, both the precautionary and the risk-adjusted growth effects reinforce each other and raise the implied option value $G(x)$, when the hedging opportunity is greater. This result implies that a larger extent of hedging delays investment when $\rho < 0$.

3.3 Comparison with the Complete Markets Solution

So far, we have assumed that the project risk cannot be spanned by the existing traded assets. If the agent can trade an additional risky asset which spans the idiosyncratic risk generated by the Brownian motion \tilde{B} , then markets are complete. Specifically, let the return of the second risky asset be given by $dS_t/S_t = rdt + \sigma_S d\tilde{B}_t$, where σ_S is a positive constant. Since the idiosyncratic risk is by definition independent of the market risk, this risky asset yields an expected rate of return r and does not demand a risk premium by the CAPM. Therefore, the implied unique

stochastic discount factor ξ is given by $-d\xi_t/\xi_t = r dt + \eta dB_t$ with $\xi_0 = 1$, where η is the Sharpe ratio of the market portfolio.

Given complete markets, standard finance theory implies that the option value and the investment threshold are independent of preferences. Indeed, we may apply the martingale method to rewrite the dynamic budget constraint as a static Arrow-Debreu budget constraint using the stochastic discount factor ξ .¹⁷ The agent's joint consumption, investment and asset allocation decision problem can then be formulated as a two-stage problem with the agent (i) choosing an investment policy to maximize the option value so that the agent's total wealth is maximized; and (ii) choosing optimal consumption given this total wealth. Using the unique stochastic discount factor ξ , we can write the option value maximization problem as follows:

$$\Phi(x) = \max_{\tau} E \left[\xi_{\tau} (X_{\tau} - I) \mid X_0 = x \right]. \quad (25)$$

As in Dixit and Pindyck (1994), $\Phi(x)$ satisfies the following differential equation:

$$r\Phi(x) = (\alpha_x - \rho\eta\sigma_x) \Phi'(x) + \frac{\sigma_x^2}{2} \Phi''(x), \quad (26)$$

and the boundary conditions $\lim_{x \rightarrow -\infty} \Phi(x) = 0$, $\Phi(x^*) = x^* - I$, and $\Phi'(x^*) = 1$. The first term on the right side of (26) is due to the fact that the correlation between the stochastic discount factor ξ and X is ρ . By a standard argument, we can derive explicit expressions for the option value and the investment threshold,

$$\Phi(x) = \frac{1}{\lambda_x} e^{\lambda_x(x-x^*)}, \quad (27)$$

$$x^* = I + \frac{1}{\lambda_x}, \quad (28)$$

where $\lambda_x = -\sigma_x^{-2}(\alpha_x - \rho\eta\sigma_x) + \sqrt{\sigma_x^{-4}(\alpha_x - \rho\eta\sigma_x)^2 + 2r\sigma_x^{-2}} > 0$.

It is important to emphasize that, unlike equation (22), there is no negative precautionary saving term in equation (26). This reflects the fact that complete markets allow the agent to perfectly diversify idiosyncratic risk. Thus, the option value under complete markets is higher than that under incomplete markets. This result also implies that a perfectly diversified agent invests later than an undiversified agent.

¹⁷See Cox and Huang (1989), and Karatzas, Lehoczky, and Shreve (1987) on the martingale method. This method and the dynamic programming method deliver the same solution. See Duffie (2001) for a textbook treatment.

4 Models with Flow Payoffs

While some real world examples may fit in the lump-sum payoff setting that we have just analyzed, there are many situations under which the investment payoffs are specified in flows over time, rather than as a lump-sum payment. As pointed out earlier, under incomplete markets, one has to derive the certainty equivalent value for the cash flows by solving the agent's consumption decision after the option exercise. We now address this issue.

In the flow payoff case, after the agent irreversibly exercises his investment option at some time τ , he obtains a perpetual stream of payoffs $\{Y_t : t \geq \tau\}$. Assume that the flow payoff process Y is governed by an arithmetic Brownian motion process:

$$dY_t = \alpha_y dt + \sigma_y dZ_t, \quad Y_0 \text{ given}, \quad (29)$$

where α_y and σ_y are positive constants and Z is a standard Brownian motion.

We present our analysis in three subsections. First, we analyze the self-insurance case in which the agent can trade only a risk-free asset, similar to Section 2. Then, we allow the agent to trade a market portfolio to partially hedge against the flow payoff risk, similar to Section 3. Finally, we allow the agent to trade an additional risky asset to span the idiosyncratic risk and compare the resulting complete markets solution with the incomplete markets solution.

4.1 Self-Insurance

When the agent can trade only a risk-free asset, the agent's wealth $\{W_t : t \geq 0\}$ after the option exercise ($\tau \leq t$) evolves according to

$$dW_t = (rW_t + Y_t - C_t) dt. \quad (30)$$

This equation resembles that in a standard incomplete markets consumption-savings model with a stream of labor income $\{Y_t : t \geq \tau\}$. At the investment time τ , the agent pays the cost I and hence wealth is lowered from $W_{\tau-}$, the level just prior to investment, to W_τ , the level immediately after the option exercise, in that $W_\tau = W_{\tau-} - I$. Before exercising the option ($0 \leq t < \tau$), the agent does not receive flow payoffs and thus his wealth evolves according to (3) as in the lump-sum case. The agent's decision problem is to choose both an investment timing strategy τ and a consumption process C so as to maximize his utility (1) subject to wealth accumulation equations (30) and (3) and a transversality condition specified in the appendix.

We solve the agent's decision problem backward by dynamic programming. Let $J(w, y)$ be the value function after the option exercise. Unlike the lump-sum payoff case, the payoff value y

is an additional state variable for J . By the standard argument, $J(w, y)$ satisfies the following HJB equation:

$$rJ(w, y) = \max_{c \in \mathbb{R}} U(c) + (rw + y - c) J_w(w, y) + \alpha_y J_y(w, y) + \frac{\sigma_y^2}{2} J_{yy}(w, y). \quad (31)$$

Without risking confusion, we still use $V(w, y)$ to denote the value function before the option exercise. Similar to Section 2, $V(w, y)$ satisfies the following HJB equation:

$$rV(w, y) = \max_{c \in \mathbb{R}} U(c) + (rw - c) V_w(w, y) + \alpha_y V_y(w, y) + \frac{\sigma_y^2}{2} V_{yy}(w, y). \quad (32)$$

We now briefly discuss the boundary conditions for the flow payoff case and relate to the lump-sum payoff case analyzed earlier. Similar to the lump-sum payoff case, the no-bubble condition $\lim_{y \rightarrow -\infty} V(w, y) = V^0(w)$ must be satisfied. Similar to, but different from the lump-sum payoff case, we have the following value matching condition:

$$V(w, y) = J(w - I, y). \quad (33)$$

This equation determines an investment boundary $\bar{y}(w)$. Moreover, the agent's optimality further requires the following smooth pasting conditions to hold:

$$\left. \frac{\partial V(w, y)}{\partial y} \right|_{y=\bar{y}(w)} = \left. \frac{\partial J(w - I, y)}{\partial y} \right|_{y=\bar{y}(w)}, \quad (34)$$

$$\left. \frac{\partial V(w, y)}{\partial w} \right|_{y=\bar{y}(w)} = \left. \frac{\partial J(w - I, y)}{\partial w} \right|_{y=\bar{y}(w)}. \quad (35)$$

Both smoothing-pasting conditions are similar to, but different from those for the lump-sum case, because the cash flow payoff y enters as an additional state variable even after the agent makes the investment.

Given CARA utility, we use a similar procedure to that in Section 2 to solve the above problem and to show that $\bar{y}(w)$ is independent of w .

Proposition 3 *The agent exercises the investment option the first time the process Y hits the threshold \bar{y} from below. After exercising the option, the optimal consumption rule is given by*

$$\bar{c}(w, y) = r(w + f(y)), \quad (36)$$

where $f(y)$ is given by

$$f(y) = \left(\frac{y}{r} + \frac{\alpha_y}{r^2} \right) - \frac{\gamma \sigma_y^2}{2r^2}. \quad (37)$$

Before exercising the option, the optimal consumption rule is given by

$$\bar{c}(w, y) = r(w + g(y)), \quad (38)$$

where (g, \bar{y}) is the solution to the following free boundary problem:

$$rg(y) = \alpha_y g'(y) + \frac{\sigma_y^2}{2} g''(y) - \frac{\gamma r \sigma_y^2}{2} g'(y)^2, \quad (39)$$

subject to the no-bubble condition $\lim_{y \rightarrow -\infty} g(y) = 0$ and the boundary conditions

$$g(\bar{y}) = f(\bar{y}) - I, \quad (40)$$

$$g'(\bar{y}) = f'(\bar{y}) = \frac{1}{r}. \quad (41)$$

Moreover, g is increasing and convex.

Unlike in the lump-sum payoff case, in the flow payoff case, the agent continues to face undiversifiable payoff risk after exercising the investment option. Thus, the agent's problem becomes a standard incomplete-markets consumption-savings problem with stochastic income $\{Y_t : t \geq \tau\}$, after exercising the investment option at time τ . Because of the CARA utility specification, we are able to derive the explicit expression for the consumption rule given in (36)-(37).¹⁸

To understand this rule, we define human wealth $h(y)$ as the present discounted value of all investment cash flows following Friedman (1957) and Hall (1978). For our arithmetic Brownian motion income process, this gives

$$h(y) \equiv E \left(\int_0^\infty e^{-rt} Y_t dt \mid Y_0 = y \right) = \frac{y}{r} + \frac{\alpha_y}{r^2}. \quad (42)$$

Note that this traditional definition of human wealth ignores the effect of risk adjustment. Using this traditional definition of human wealth, we may rewrite the consumption rule given in (36) and (37) as follows:

$$\bar{c}(w, y) = r \left(w + h(y) - \frac{\gamma \sigma_y^2}{2r^2} \right). \quad (43)$$

When $\gamma = 0$ or $\sigma_y = 0$, the consumption rule (43) is simply given by $r(w + h(y))$, the annuity value of the sum of financial wealth w and human wealth $h(y)$. This is Friedman's permanent-income hypothesis. Importantly, when the agent has precautionary motive ($\gamma > 0$) and when markets are incomplete, a precautionary savings demand arises. This demand is given by $\gamma \sigma_y^2 / (2r^2)$, which increases with precautionary motive γ and volatility σ_y . We can follow a

¹⁸Caballero (1991b) derives this consumption rule in discrete time. Wang (2004) extends Caballero (1991b) to more general bi-variate income processes with partial observability and hence increases precautionary savings demand due to estimation risk. Wang(2003) shows that general equilibrium restriction eliminates the precautionary saving demand in Caballero (1991b).

similar method as in Section 2.3 and interpret $f(y)$ as the certainty equivalent (risk-adjusted) human wealth. Since $f(y) = h(y) - \gamma\sigma_y^2 / (2r^2)$, the certainty equivalent human wealth $f(y)$ decreases in γ and also in volatility σ_y .

Although the differential equation (39) is similar to its counterpart (10) for the lump-sum payoff case, we emphasize that the boundary conditions for the flow payoff case are different from those for the lump-sum payoff case in Proposition 1. Since the agent receives a stream of stochastic flow payoffs $\{Y_t : t \geq \tau\}$ after exercising the option at τ , he values this stream of payoffs with the certainty equivalent human wealth $f(y)$ given in (37). Similar to Section 2.3, we may use the certainty equivalent approach in the literature on the pricing of nontraded assets to interpret $f(y)$ as the implied project value associated with the investment option.

Because of the extra effect of risk measured by σ_y and risk attitude measured by γ on the implied project value $f(y)$, σ_y and γ may have different impacts on investment timing in the flow payoff case than in the lump-sum payoff case.

Effects of Risk and Risk Attitude We analyze the impact of γ and σ_y on the implied option value $g(y)$ and on the investment threshold \bar{y} via the approximation method. We approximate the implied option value $g(y)$ and the investment threshold \bar{y} simultaneously to the first order of σ_y^2 . We then obtain the approximate investment threshold:

$$\bar{y}_1 = \bar{y}_0 + \frac{1}{2\alpha_y}\sigma_y^2, \quad (44)$$

where $\bar{y}_0 = rI$ is the exactly solved investment threshold in the deterministic case ($\sigma_y = 0$). Thus, to the first-order approximation, the investment threshold \bar{y}_1 increases in volatility as the standard real option theory predicts.

Equation (44) also reveals that, to the first-order approximation, the agent's risk attitude does not affect investment timing. This result differs from the lump-sum payoff case, in which the investment threshold is lowered by the agent's precautionary motive to the first-order approximation. The intuition for this difference is as follows. Unlike in the lump-sum case, the agent receives a stream of uninsurable payoffs after the option exercise in the flow payoff case. Therefore, the agent's precautionary motive lowers not only the implied option value $g(y)$, but also the implied project value $f(y)$. These precautionary effects on $g(y)$ and $f(y)$ cancel out to the first-order approximation.

To further understand the impact of the agent's precautionary motive γ on the investment decision, we use the second-order approximation with respect to σ_y^2 and obtain the following

approximate investment threshold:

$$\bar{y}_2 = \bar{y}_1 + \frac{1}{\alpha_y^2} \left(\gamma - \frac{r}{2\alpha_y} \right) \sigma_y^4, \quad (45)$$

where \bar{y}_1 is given in (44). Equation (45) indicates that, to the second-order approximation, the investment threshold increases in γ . While the precautionary saving effect is present both before and after the option exercise as argued earlier, the precautionary saving effect, to the second-order approximation, has a larger impact on $f(y)$ than on $g(y)$. The intuition is as follows. Before exercising the option, the agent may time when to invest in the risky investment. This additional flexibility before exercising the option on the margin lowers the project risk. Thus, the precautionary saving effect is, to the second-order approximation, stronger after exercising the option than before. This suggests that an increase in the precautionary motive γ lowers $f(y)$ more than $g(y)$, thereby delaying the exercise of the option.

Finally, we use numerical solutions to conduct further analysis. Figure 4 plots the investment threshold as a function of volatility σ_y and the parameter γ . This figure confirms our preceding approximation results. Moreover, it illustrates that the effects of volatility σ_y on the investment threshold are stronger when the agent is more precautionary, i.e., when γ is higher.

[Insert Figure 4 Here]

Figure 5 illustrates the effect of changes in γ . An increase in γ raises precautionary savings both after and before the option exercise, thereby lowering both the implied project value $f(y)$ and the implied option value $g(y)$. This figure confirms our earlier analysis that $f(y)$ is lowered more than $g(y)$, so that the agent delays exercising the investment option.

[Insert Figure 5 Here]

4.2 Hedging

We now turn to the flow payoff case with hedging opportunity. We will focus on how the hedging opportunity affects the agent's investment and consumption decisions and highlight the difference between the flow payoff case and the lump-sum payoff case.

Let π_t denote the amount allocated in the risky asset with returns given in (15) at time t (in units of the consumption good). Before the agent exercises the investment option at time τ , his wealth accumulation is the same as (17). After time τ , his wealth evolves as follows:

$$dW_t = [rW_t + \pi_t(\mu_e - r) + Y_t - C_t] dt + \pi_t \sigma_e dB_t. \quad (46)$$

Note that the flow payoff Y appears in (46), not in (17). As before, the agent's wealth immediately after his investment W_τ is given by $W_\tau = W_{\tau-} - I$, where $W_{\tau-}$ denotes his wealth level just prior to his investment at time τ . The following proposition characterizes the solution.

Proposition 4 *The agent exercises the investment option the first time the process Y hits the threshold \bar{y} from below. After exercising the option, the optimal consumption and portfolio rules are given by*

$$\bar{c}(w, y) = r \left(w + f(y) + \frac{\eta^2}{2\gamma r^2} \right), \quad (47)$$

$$\bar{\pi}(w, y) = \frac{\eta}{\gamma \sigma_e} \frac{1}{r} - \frac{\rho \sigma_y}{\sigma_e r}, \quad (48)$$

where $f(y)$ is given by

$$f(y) = \left(\frac{1}{r} y + \frac{\alpha_y - \rho \eta \sigma_y}{r^2} \right) - \frac{\gamma \sigma_y^2 (1 - \rho^2)}{2r^2}. \quad (49)$$

Before exercising the option, the optimal consumption and portfolio rules are given by

$$\bar{c}(w, y) = r \left(w + g(y) + \frac{\eta^2}{2\gamma r^2} \right), \quad (50)$$

$$\bar{\pi}(w, y) = \frac{\eta}{\gamma \sigma_e} \frac{1}{r} - \frac{\rho \sigma_y}{\sigma_e} g'(y), \quad (51)$$

where (g, \bar{y}) is the solution to the following free boundary problem:

$$r g(y) = (\alpha_y - \rho \eta \sigma_y) g'(y) + \frac{\sigma_y^2}{2} g''(y) - \frac{\gamma r \sigma_y^2 (1 - \rho^2)}{2} g'(y)^2, \quad (52)$$

subject to the no-bubble condition $\lim_{y \rightarrow -\infty} g(y) = 0$, and the boundary conditions

$$g(\bar{y}) = f(\bar{y}) - I, \quad (53)$$

$$g'(\bar{y}) = \frac{1}{r}. \quad (54)$$

Moreover, g is increasing and convex.

When $\rho = 0$, the solution for (g, \bar{y}) reduces to that without hedging opportunities described in Proposition 3. As in Section 4.1, we can use the certainty equivalent approach to interpret $f(y)$ as the implied project value and $g(y)$ as the implied option value. Unlike the lump-sum payoff case with hedging opportunity in Section 3, hedging also influences $f(y)$. As in Section 3, there are two effects: the precautionary saving effect and the risk-adjusted growth effect. The precautionary saving effect is mitigated by the hedging opportunity since hedging reduces

the agent's exposure to idiosyncratic risk from σ_y to $\sqrt{1 - \rho^2} \sigma_y$. Thus, hedging raises $f(y)$. On the other hand, hedging also changes $f(y)$ by lowering (raising) the growth rate of the payoff process from α_y to $(\alpha_y - \rho\eta\sigma_y)$ when $\rho > 0$ ($\rho < 0$), because the agent holds a short (long) position in the risky asset. Equation (48) reveals that the hedging demand after the option exercise is independent of the state of the investment opportunities (Merton (1969)). This is in contrast to the hedging demand before the option exercise given in (51), which depends on the moneyness of the investment option as in Section 3.

Because the extent of hedging opportunity influences not only the implied option value $g(y)$, but also the implied project value $f(y)$, hedging may have a different effect on the investment threshold, compared with the lump-sum payoff case in Section 3, for a similar reason to that discussed in Section 4.1. We illustrate this point in Figure 6. Figure 6a indicates that when γ is sufficiently large, hedging raises both the implied option value and the implied project value due to the stronger precautionary saving effect. Since the implied project value is raised more than the implied option value, the agent exercises the investment option earlier. Figure 6b indicates that when γ is sufficiently small, hedging lowers the implied option value and the implied project value due to the stronger risk-adjusted growth effect. Since the implied project value is lowered more, the agent delays the option exercise.

[Insert Figure 6 Here]

4.3 Comparison with the Complete Markets Solution

Similar to Section 3.3, suppose that the agent can trade an additional risky asset which spans the idiosyncratic risk. Then markets are complete. We can use the same unique stochastic discount factor ξ derived in Section 3.3 to value the investment option as follows:

$$\Psi(y) = \max_{\tau} E \left[\int_{\tau}^{\infty} \xi_t Y_t dt - \xi_{\tau} I \middle| Y_0 = y \right]. \quad (55)$$

By a standard argument, $\Psi(y)$ satisfies the following differential equation:

$$r\Psi(y) = (\alpha_y - \rho\eta\sigma_y) \Psi'(y) + \frac{\sigma_y^2}{2} \Psi''(y), \quad (56)$$

and the boundary conditions $\lim_{y \rightarrow -\infty} \Psi(y) = 0$, $\Psi(y^*) = F(y^*) - I$, and $\Psi'(y^*) = 1/r$, where

$$F(y) = \frac{1}{r}y + \frac{\alpha_y - \rho\eta\sigma_y}{r^2}. \quad (57)$$

We derive the following explicit expressions for the option value and the investment threshold:

$$\Psi(y) = \frac{1}{r\lambda_y} e^{\lambda_y(y-y^*)}, \quad (58)$$

$$y^* = rI - \frac{\alpha_y - \rho\eta\sigma_y}{r} + \frac{1}{\lambda_y}, \quad (59)$$

where $\lambda_y = -\sigma_y^{-2} (\alpha_y - \rho\eta\sigma_y) + \sqrt{\sigma_y^{-4} (\alpha_y - \rho\eta\sigma_y)^2 + 2r\sigma_y^{-2}} > 0$.

We observe that, under complete markets, the lump-sum and flow payoff formulations are mathematically equivalent, since we may discount cash flows using the unique stochastic discount factor ξ . Specifically, by defining $X_t = \xi_t^{-1} E_t \left(\int_t^\infty \xi_s Y_s ds \right) = F(Y_t)$, we can show that the problems (25) and (55) are equivalent. Thus, they deliver the same option value $\Phi(x) = \Psi(y)$ and investment timing strategy. However, this equivalence fails when the investment opportunity is not tradable and not spanned by the existing traded assets.

In the flow payoff case, the negative precautionary saving terms vanish in the expressions for both the option value (56) and the project value (57) under complete markets. Therefore, the elimination of precautionary savings raises both the option value and the project value. Following similar analyses and intuition for the self-insurance model with flow payoffs in Section 4.1, we can show that the project value is raised more than the option value. Therefore, we conclude that the agent invests sooner under complete markets than under incomplete markets. This result is opposite to that in the lump-sum payoff case analyzed in Section 3.3.

5 Conclusions

Entrepreneurs' business investment opportunities are often nontradable and their payoffs cannot be spanned by existing traded assets due to reasons such as incentives and informational asymmetries. These features invalidate the standard real options approach to investment. Extending this approach, we provide a utility-based model to analyze an agent's real investment, consumption, and portfolio choice decisions.

We show that project volatility has not only a positive option effect, but also a negative effect on the implied option value. The latter effect is induced by the precautionary saving motive. For the lump-sum payoff case, risk aversion accelerates investment. Unlike the standard real options analysis, an increase in project volatility may accelerate investment if the agent has a sufficiently strong precautionary motive. We further extend our model to allow for the opportunity to hedge. We show that the agent's hedging demand increases when he is closer to exercising his investment option. Moreover, hedging affects investment decisions by changing the expected growth of wealth and reducing the agent's exposure to idiosyncratic risk. We also analyze the flow payoff case. Unlike the standard real options analysis, the lump-sum and flow payoff cases have different model implications. Because the precautionary saving effect matters both before and after investment in the flow payoff case, many predictions in this case differ from and may even be opposite to those in the lump-sum payoff case.

Our model has empirical implications. For example, our model suggests that a positive investment-uncertainty relationship may potentially arise for entrepreneurship. Thus, one may be cautious in interpreting some conflicting results found in empirical studies.¹⁹ Our model also suggests that the investment behavior of undiversified individuals is different from that of well-diversified individuals or institutions. In particular, risk attitude plays an important role under incomplete markets. Consider again the real estate development example. Suppose that we have a sample containing both undiversified individual developers and publicly traded real estate investment trusts (REITs). Suppose that both individual entrepreneurs and REITs specialize in development of and not management of the properties. Then our model predicts that the individual entrepreneurs are more likely to develop earlier than the publicly traded REITs, because the implied option value of waiting is lower for individual developers. However, if they also manage the properties after completion of development, then the preceding prediction is reversed because the properties are also less valuable to the undiversified individual developers.

In order to analyze the effect of incomplete markets on investment in the simplest possible setting, we have intentionally ignored the wealth effect by adopting the CARA utility. However, the wealth effect may potentially play an important role in the determinants of entrepreneurship. We extend our analysis to incorporate the wealth effect on entrepreneurial investment in Miao and Wang (2005a). Finally, when entrepreneurs invest in nontradable projects, they often need to make financing decisions jointly. We analyze the interaction between investment and financing decisions in Miao and Wang (2005b).

¹⁹See Quigg (1993), Berger et al. (1996), Leahy and White (1996), and Moel and Tufano (1998) for empirical works. See Caballero (1991a) for a theoretical analysis.

Appendices

A Proofs

Proof of Proposition 1: We can easily show that the value function after the option exercise is given by $V^0(w) = -\frac{1}{\gamma r} \exp(-\gamma r w)$. We conjecture that the value function before the option exercise takes the following form:

$$V(w, x) = -\frac{1}{\gamma r} \exp[-\gamma r (w + G(x))]. \quad (\text{A.1})$$

From the first-order condition $U'(c) = V_w(w, x)$, we can derive the consumption policy before the option exercise given in (9). Substituting it into the HJB equation (5), we can show that $G(x)$ satisfies the ODE (10). Given the functional forms of the value functions, we can also show that the no-bubble condition, the value-matching and the smooth-pasting conditions become the boundary conditions in Proposition 1. By a standard dynamic programming argument, one can show that V satisfies

$$V(w, x) = \max_{(\tau, C)} E \left[\int_0^\tau e^{-rt} U(C_t) dt + e^{-r\tau} V^0(W_\tau + X_\tau - I) \mid (W_0, X_0) = (w, x) \right]. \quad (\text{A.2})$$

Consider $x < x'$. For $X_0 = x'$, let τ' be the optimal investment time and $\{C'_t : 0 \leq t \leq \tau'\}$ be the optimal consumption process before investment. Since V^0 is an increasing function and, given any sample path,

$$X_{\tau'} \equiv x + \alpha_x \tau' + \sigma_x W_{\tau'} < X'_{\tau'} \equiv x' + \alpha_x \tau' + \sigma_x W_{\tau'},$$

we have

$$\int_0^{\tau'} e^{-rt} U(C'_t) dt + e^{-r\tau'} V^0(W_{\tau'} + X_{\tau'} - I) < \int_0^{\tau'} e^{-rt} U(C'_t) dt + e^{-r\tau'} V^0(W_{\tau'} + X'_{\tau'} - I).$$

Taking conditional expectations yields

$$\begin{aligned} & E \left[\int_0^{\tau'} e^{-rt} U(C'_t) dt + e^{-r\tau'} V^0(W_{\tau'} + X_{\tau'} - I) \mid (W_0, X_0) = (w, x) \right] \\ & < E \left[\int_0^{\tau'} e^{-rt} U(C'_t) dt + e^{-r\tau'} V^0(W_{\tau'} + X'_{\tau'} - I) \mid (W_0, X_0) = (w, x') \right] = V(w, x'). \end{aligned}$$

Given the wealth dynamics described in Section 2.1, $\{C'_t : 0 \leq t \leq \tau'\}$ and τ' are also feasible for $X_0 = x$. Thus, the left side of the above equation is less or equal to $V(w, x)$ by (A.2). So, $V(w, x) < V(w, x')$ and V is increasing in x . Thus, the optimal investment policy is characterized by the trigger policy described in the proposition. We can also use a similar method to prove V is concave in x using the fact that V^0 is concave in x . Finally, given the functional form of V in (A.1), we can show that G is increasing and convex in x . Q.E.D.

Proof of Proposition 2: Without risk of confusion, we still use $V^0(w)$ and $V(w, x)$ to denote the value function after and before the option exercise, respectively, when the agent can trade a risky asset. By a standard argument, V^0 satisfies the following HJB equation:

$$rV^0(w) = \max_{(c, \pi) \in \mathbb{R}^2} U(c) + [rw + \pi(\mu_e - r) - c] V_w^0(w) + \frac{(\pi\sigma_e)^2}{2} V_{ww}^0(w). \quad (\text{A.3})$$

The transversality condition $\lim_{T \rightarrow \infty} E[e^{-rT} V^0(W_T)] = 0$ must also be satisfied. Given CARA utility, one can follow Merton (1969) to derive the consumption and portfolio rules in (18)-(19) and

$$V^0(w) = -\frac{1}{\gamma r} \exp\left[-\gamma r \left(w + \frac{\eta^2}{2\gamma r^2}\right)\right]. \quad (\text{A.4})$$

Before the option exercise, the value function $V(w, x)$ satisfies the following HJB equation:

$$\begin{aligned} rV(w, x) = & \max_{(c, \pi) \in \mathbb{R}^2} U(c) + [rw + \pi(\mu_e - r) - c] V_w(w, x) + \alpha_x V_x(w, x) \\ & + \frac{\sigma_x^2}{2} V_{xx}(w, x) + \frac{(\pi\sigma_e)^2}{2} V_{ww}(w, x) + \pi\sigma_e\sigma_x\rho V_{wx}(w, x). \end{aligned} \quad (\text{A.5})$$

We conjecture that the value function V takes the form

$$V(w, x) = -\frac{1}{\gamma r} \exp\left[-\gamma r \left(w + G(x) + \frac{\eta^2}{2\gamma r^2}\right)\right], \quad (\text{A.6})$$

where $G(x)$ is a function to be determined. Using the first-order conditions,

$$U'(c) = V_w(w, x), \quad \pi = \frac{-V_w(w, x) \mu_e - r}{V_{ww}(w, x) \sigma_e^2} + \frac{-V_{wx}(w, x) \rho \sigma_x}{V_{ww}(w, x) \sigma_e}, \quad (\text{A.7})$$

one can derive the optimal consumption and portfolio policies before exercising the option given in (20)-(21). Plugging these expressions back into the HJB equation gives (22). As in Section 2, the boundary conditions are given by the no-bubble, value-matching, and smooth-pasting conditions similar to (6)-(8). Using these boundary conditions, one can derive the boundary conditions in Proposition 2. The rest of the proof follows a similar argument to that in Proposition 1. Q.E.D.

Proof of Proposition 3: We conjecture that the value function after the option exercise J takes the following form:

$$J(w, y) = -\frac{1}{\gamma r} \exp[-\gamma r (w + f(y))], \quad (\text{A.8})$$

where $f(y)$ is a function to be determined. To solve for this function, we use the first-order condition $U'(c) = J_w(w, y)$ to derive the optimal consumption rule given in (36). Substitute it

back into the HJB equation (31) to derive the following ODE:

$$0 = (y - rf(y)) + \alpha_y f'(y) + \frac{\sigma_y^2}{2} [f''(y) - \gamma r f'(y)^2]. \quad (\text{A.9})$$

It can be verified that its solution is given by (37). Moreover, it is such that the value function satisfies the transversality condition $\lim_{T \rightarrow \infty} E [e^{-rT} J(W_T, Y_T)] = 0$.

We conjecture that the value function before the option exercise, $V(w, y)$, takes the form:

$$V(w, y) = -\frac{1}{\gamma r} \exp[-\gamma r (w + g(y))], \quad (\text{A.10})$$

where $g(y)$ is a function to be determined. From the first-order condition $U'(c) = V_w(w, y)$, we can derive the consumption policy before investment given in (38). Substituting it into the HJB equation (32), we can show that $g(y)$ satisfies the ODE (39). By a standard dynamic programming argument, one can show that V satisfies

$$V(w, y) = \max_{\tau, C} E \left[\int_0^\tau e^{-rt} U(C_t) dt + e^{-r\tau} J(W_\tau - I, Y_\tau) \mid (W_0, Y_0) = (w, y) \right]. \quad (\text{A.11})$$

Since it follows from (A.8) that J is increasing and concave in y , one can show that V is also increasing and concave in y . The rest of the proof follows from a similar argument to that in Proposition 1. Q.E.D.

Proof of Proposition 4: Without risk of confusion, we still use $J(w, y)$ and $V(w, y)$ to denote the value function after and before the option exercise, respectively, when the agent can also trade a risky asset. By a standard argument, $J(w, y)$ satisfies the HJB equation

$$\begin{aligned} rJ(w, y) = & \max_{(c, \pi) \in \mathbb{R}^2} U(c) + [rw + \pi(\mu_e - r) + y - c] J_w(w, y) + \alpha_y J_y(w, y) \quad (\text{A.12}) \\ & + \frac{\sigma_y^2}{2} J_{yy}(w, y) + \frac{(\pi \sigma_e)^2}{2} J_{ww}(w, y) + \pi \sigma_e \sigma_y \rho J_{wy}(w, y). \end{aligned}$$

The transversality condition $\lim_{T \rightarrow \infty} E [e^{-rT} J(W_T, Y_T)] = 0$ must also be satisfied. We conjecture that $J(w, y)$ takes the following form:

$$J(w, y) = -\frac{1}{\gamma r} \exp \left[-\gamma r \left(w + f(y) + \frac{\eta^2}{2\gamma r^2} \right) \right], \quad (\text{A.13})$$

where the function f is to be determined. By the first-order conditions,

$$U'(c) = J_w(w, y), \quad \pi = \frac{-J_w(w, y) \mu_e - r}{J_{ww}(w, y) \sigma_e^2} + \frac{-J_{wy}(w, y) \rho \sigma_y}{J_{ww}(w, y) \sigma_e}, \quad (\text{A.14})$$

one can derive the optimal consumption and portfolio policies after investment given in (50)-(51). Substituting them back into the HJB equation (A.12), one can derive the solution for $f(y)$ given in (49). It can be verified that this solution satisfies the transversality condition.

The value function before the option exercise, V , satisfies the following HJB equation:

$$\begin{aligned} rV(w, y) = & \max_{(c, \pi) \in \mathbb{R}^2} U(c) + [rw + \pi(\mu_e - r) - c] V_w(w, y) + \alpha_y V_y(w, y) \\ & + \frac{\sigma_y^2}{2} V_{yy}(w, y) + \frac{(\pi\sigma_e)^2}{2} V_{ww}(w, y) + \pi\sigma_e\sigma_y\rho V_{wy}(w, y). \end{aligned} \quad (\text{A.15})$$

We conjecture that the value function V takes the following form:

$$V(w, y) = -\frac{1}{\gamma r} \exp \left[-\gamma r \left(w + g(y) + \frac{\eta^2}{2\gamma r^2} \right) \right], \quad (\text{A.16})$$

where $g(y)$ is a function to be determined. Using the first-order conditions,

$$U'(c) = V_w(w, y), \quad \pi = \frac{-V_w(w, y) \mu_e - r}{V_{ww}(w, y) \sigma_e^2} + \frac{-V_{wy}(w, y) \rho \sigma_y}{V_{ww}(w, y) \sigma_e}, \quad (\text{A.17})$$

one can derive the optimal consumption and portfolio policies before investment given in (47)-(48). Plugging these expressions into the HJB equation gives a differential equation for $g(\cdot)$. The rest of the proof follows from a similar argument to that in Propositions 1 and 3. Q.E.D.

B Approximation Method

In this appendix, we provide our approximation solution methodology. We sketch out the procedure for the self insurance model with a lump-sum payoff. Essentially identical procedures may be applied to models in Section 3 and 4. We may divide the procedure into four steps.

Step 1. Solve for the case with deterministic payoff ($\sigma_x^2 = 0$). With $\sigma_x = 0$, risk attitude (γ) does not affect the investment threshold. The implied option value $G_0(x)$ and the investment threshold \bar{x}_0 are both known in closed form and are given by

$$G_0(x) = \frac{\alpha_x}{r} \exp \left[\frac{r}{\alpha_x} (x - \bar{x}_0) \right], \quad x \leq \bar{x}_0, \quad (\text{B.1})$$

$$\bar{x}_0 = I + \frac{\alpha_x}{r}. \quad (\text{B.2})$$

Step 2. Consider small σ_x^2 . Conjecture that the approximate option value and the investment threshold are

$$G(x) \approx G_0(x) + G_1(x) \sigma_x^2, \quad (\text{B.3})$$

$$\bar{x}_1 = \bar{x}_0 + \delta_1 \sigma^2, \quad (\text{B.4})$$

where $G_0(x)$ and \bar{x}_0 are solved in Step 1, and $G_1(x)$ and δ_1 are the coefficient function and the coefficient to be determined.

Step 3. Plugging the approximate solution (B.3) into the ODE (10) and boundary conditions (11)-(12) and keeping the terms up to σ_x^2 , we have the following:

$$\alpha_x G_1'(x) + \frac{1}{2} G_0''(x) - \frac{\gamma r}{2} G_0'(x)^2 = r G_1(x), \quad (\text{B.5})$$

subject to $G_1(\bar{x}_1) = 0$ and $G_1'(\bar{x}_1) = -r\delta_1/\alpha_x$. Note that unlike the original nonlinear ODE (10) for $G(x)$, we now have a free boundary problem defined by a first-order ODE (B.5) for $G_1(x)$ with certain boundary conditions.

Step 4. Solving the above differential equation gives our reported solution in (14) and

$$G_1(x) = \frac{r}{2\alpha_x^2} (\bar{x}_0 - x) e^{-\frac{r}{\alpha_x}(\bar{x}_0 - x)} - \frac{\gamma}{2} \left[e^{-\frac{r}{\alpha_x}(\bar{x}_0 - x)} - e^{-\frac{2r}{\alpha_x}(\bar{x}_0 - x)} \right], x \leq \bar{x}_1.$$

C Computation Method

We describe the solution method to the free boundary problem described in Proposition 3. The problems described in other propositions can be solved similarly. We use the projection method implemented with collocation (Judd (1998)). We do not use the traditional shooting method or finite difference method because these methods are inefficient for our nonlinear problem and extensive simulations.

We first rewrite the second order ODE (39) as a system of first-order ODEs. Let $\Delta(y) = g'(y)$. Then (39) can be rewritten as

$$\Delta'(y) = \frac{2}{\sigma_y^2} (rg(y) - \alpha_y \Delta(y)) + \gamma r \Delta(y)^2. \quad (\text{C.1})$$

The boundary conditions are

$$\lim_{y \rightarrow -\infty} g(y) = 0, \quad (\text{C.2})$$

$$g(\bar{y}) = f(\bar{y}) - I, \quad (\text{C.3})$$

$$\Delta(\bar{y}) = 1/r. \quad (\text{C.4})$$

Note that condition (C.2) states that when y goes to minus infinity, the agent never exercises the investment option, and hence the implied option value is equal to zero.

The idea of the algorithm is to first ignore the smooth-pasting condition (C.4) and then to solve a two point boundary value problem with a guessed threshold value \bar{y}_0 . Since the boundary condition (C.2) is open ended, we pick a very small negative number \underline{y} and set $g(\underline{y}) = 0$. The true value of the threshold is found by adjusting \bar{y}_0 so that the smooth-pasting condition (C.4) is satisfied. We then adjust \underline{y} so that the solution is not sensitive to this value. The algorithm is outlined as follows.

Step 1. Start with a guess \bar{y}_0 and a preset order n .

Step 2. Use Chebyshev polynomial to approximate g and Δ :

$$g(y; a) = \sum_{i=0}^n a_i T_i(y), \quad \Delta(y; b) = \sum_{i=0}^n b_i T_i(y), \quad (\text{C.5})$$

where $T_i(y)$ is the Chebyshev polynomial of order i , and $a = (a_0, a_1, \dots, a_n)$ and $b = (b_0, b_1, \dots, b_n)$ are $2n + 2$ constants to be determined. Substitute the above expressions into the preceding system of ODEs and evaluate it at n roots of $T_n(y)$. Together with the two boundary conditions, we then have $2n + 2$ equations for $2n + 2$ unknowns $a = (a_0, a_1, \dots, a_n)$ and $b = (b_0, b_1, \dots, b_n)$. Let the solution be \hat{a} and \hat{b} .

Step 3. Search for \bar{y}_0 such that the smooth-pasting condition, $\Delta(\bar{y}_0; \hat{b}) = 1/r$, is approximately satisfied.

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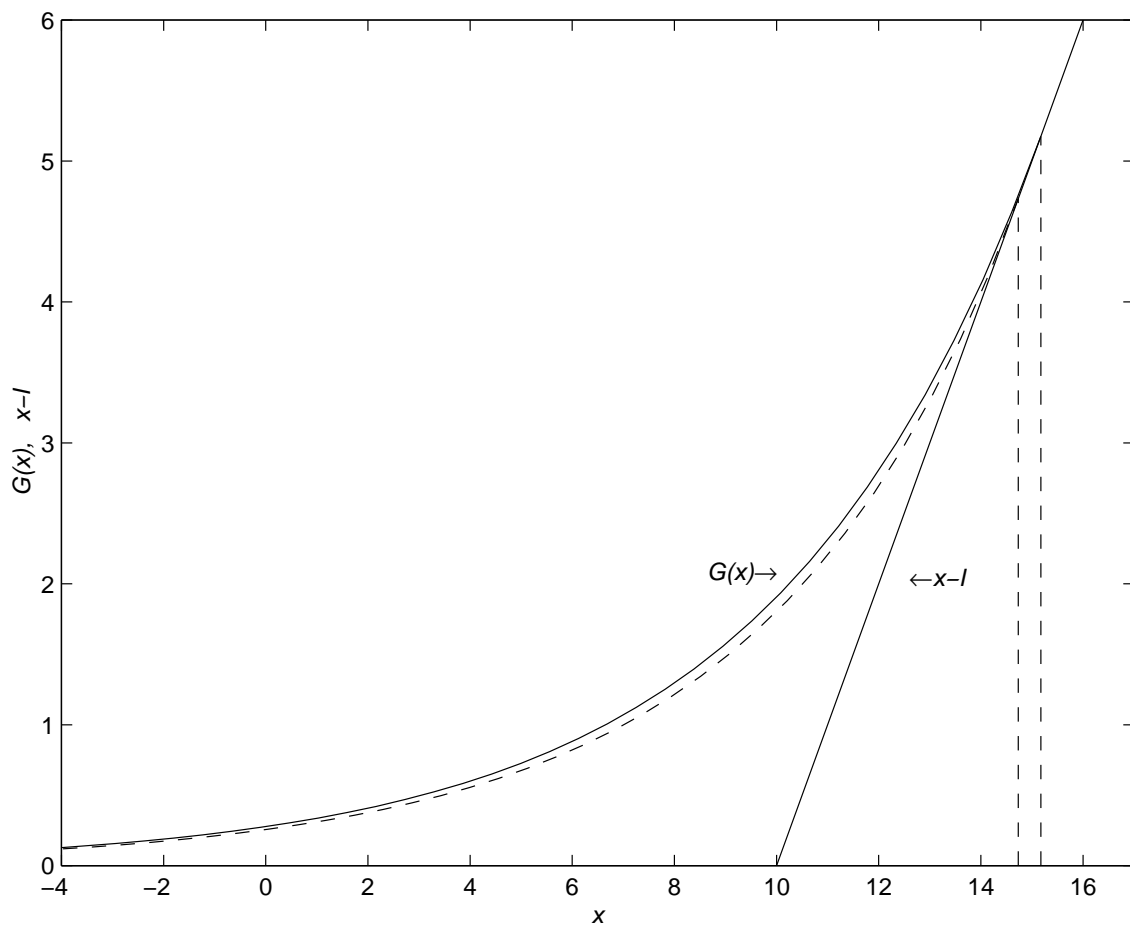


Figure 1: **Implied option value $G(x)$.** This figure plots the functions $x - I$ and $G(x)$ for the model in Section 2. The parameter values are set as follows: $r = 2\%$, $\alpha_x = 0.1$, $\sigma_x = 20\%$, and $I = 10$. The solid curve is for $\gamma = 1$, and the dashed curve is for $\gamma = 25$.

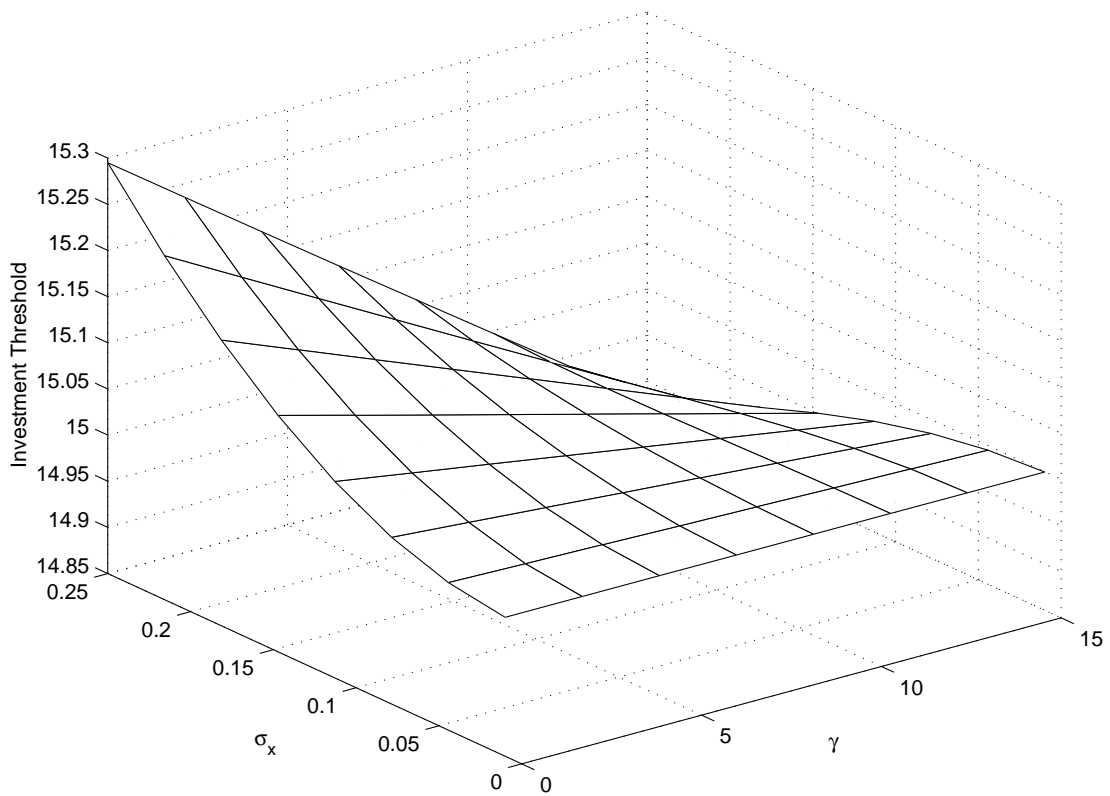


Figure 2: **Investment threshold, risk aversion, and project volatility.** This figure plots the investment threshold at varying levels of γ and σ_x for the lump sum payoff case. Other parameter values are set as $r = 2\%$, $\alpha_x = 0.1$, and $I = 10$.

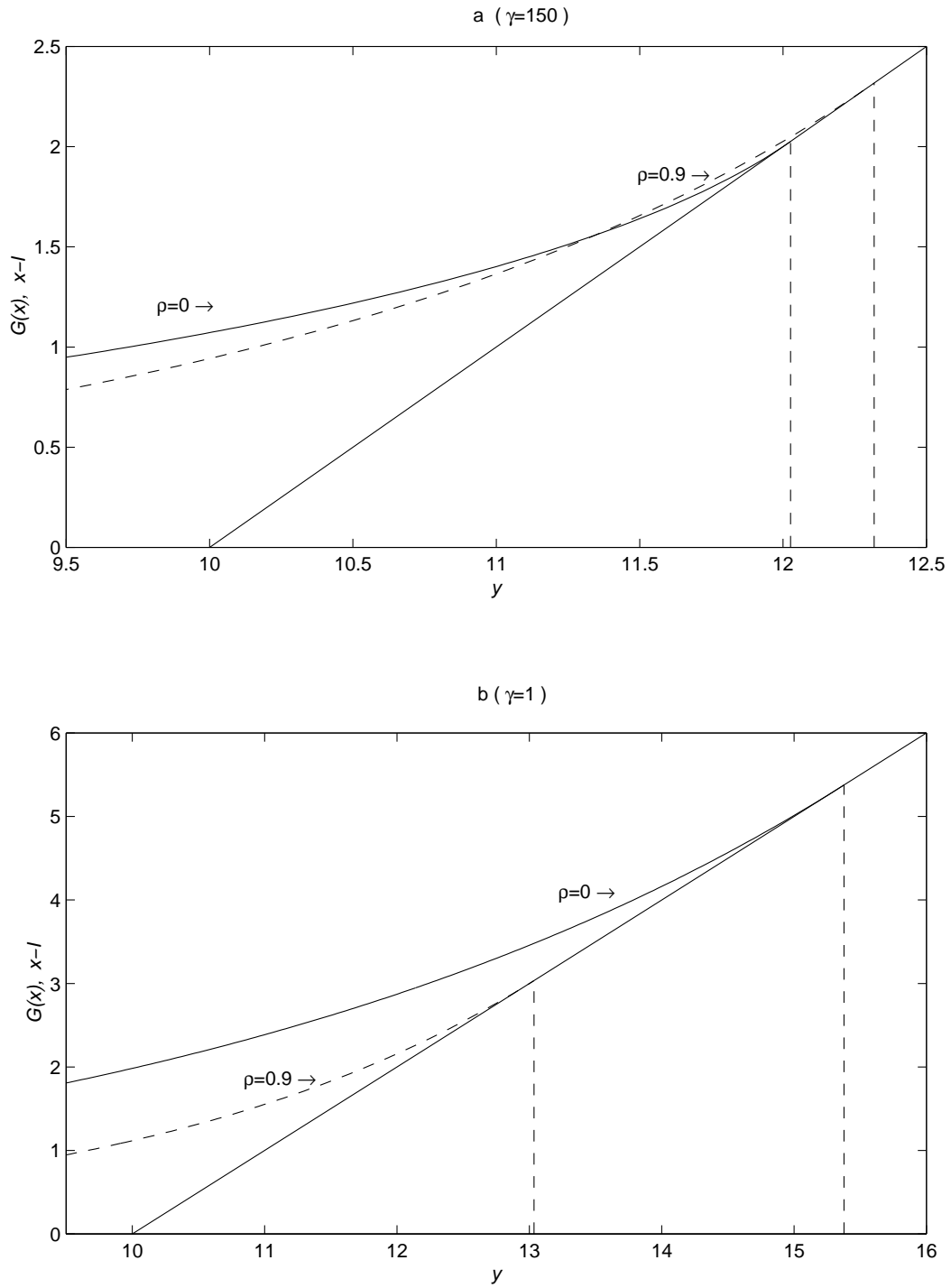


Figure 3: **Effects of hedging in the lump sum payoff case.** The dashed curves plot $G(x)$ for $\rho = 0.9$, and the solid curves plot $G(x)$ for $\rho = 0$. The top figure is for $\gamma = 150$, and the bottom figure is for $\gamma = 1$. Other parameter values are set as $r = 2\%$, $\alpha_x = 0.1$, $\sigma_x = 30\%$, $\eta = 0.2$, and $I = 10$.

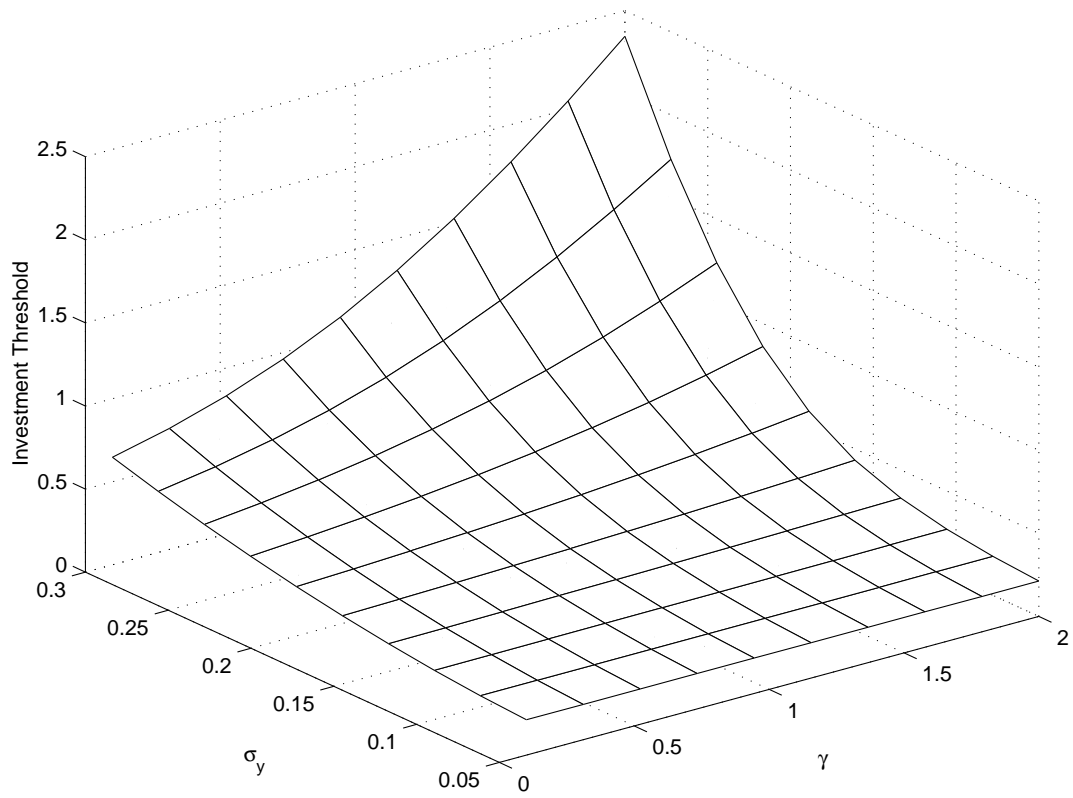


Figure 4: **Investment threshold, risk aversion, and project volatility.** This figure plots the investment threshold at varying levels of γ and σ_y for the flow payoff case. Other parameter values are set as $\beta = r = 2\%$, $\alpha_y = 0.1$, and $I = 10$

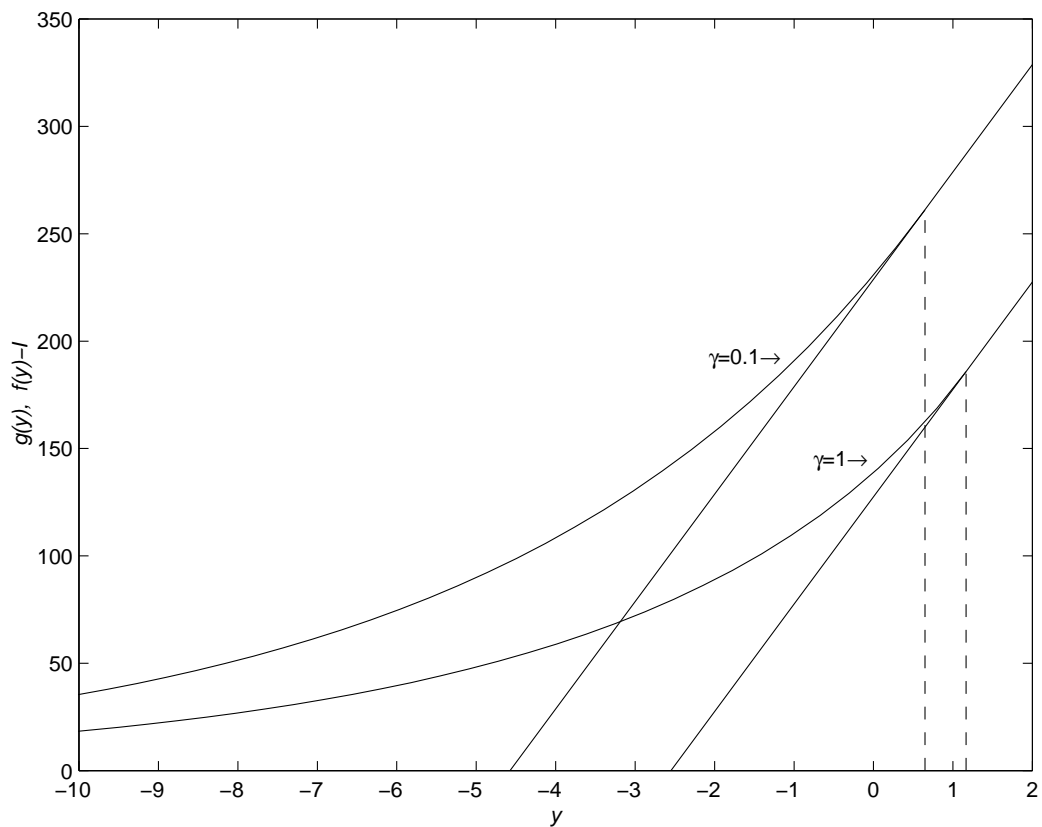


Figure 5: **Impact of changes in γ in the flow payoff case.** This figure plots the impact on $g(y)$ and $f(y)$ when the value of γ is increased from 1 to 2. Other parameter values are set as $\beta = r = 2\%$, $\alpha_y = 0.1$, $\sigma_y = 30\%$, and $I = 10$.

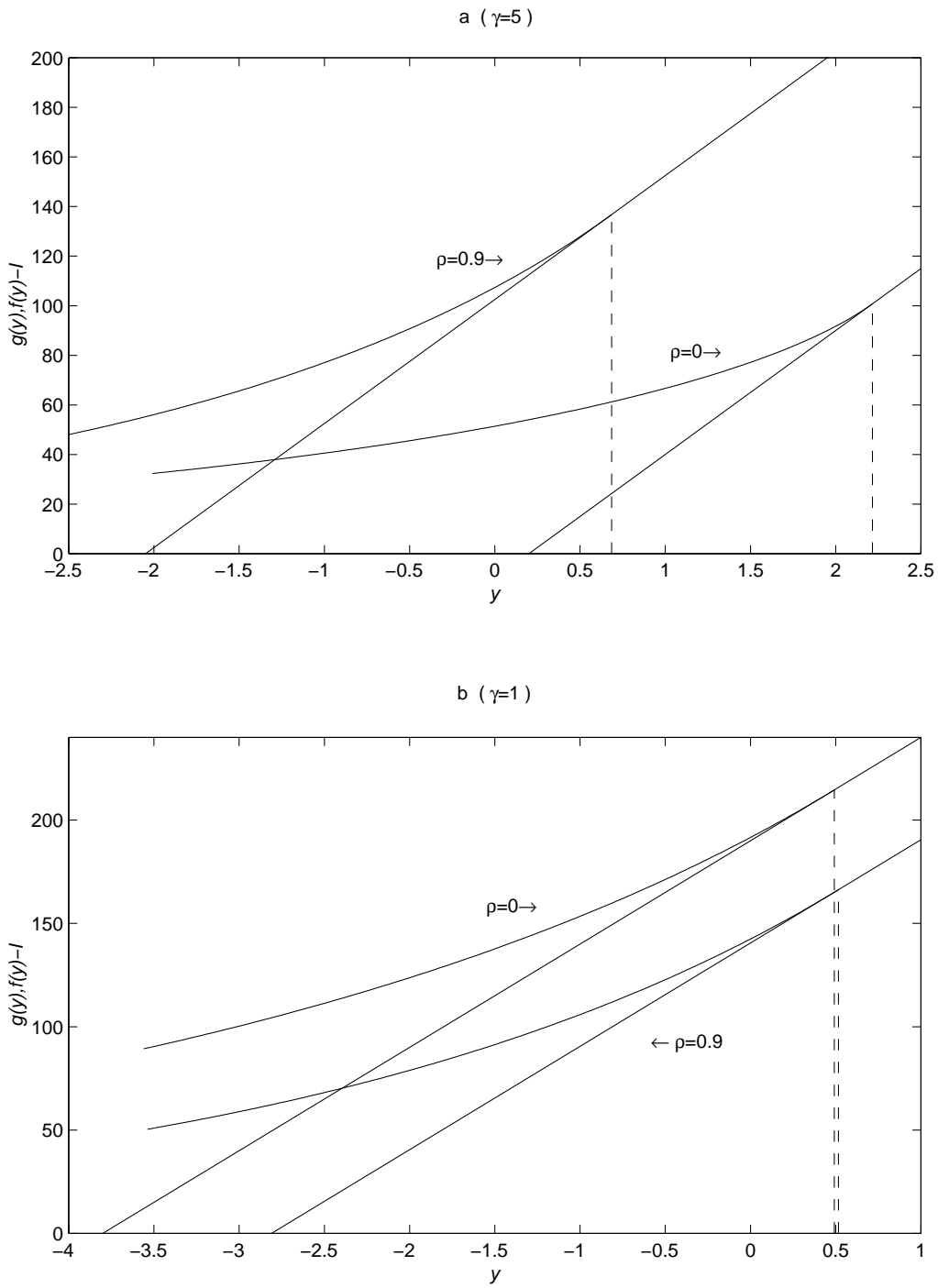


Figure 6: **Impact of changes in the correlation in the flow payoff case.** These figures plot the changes of functions $f(y) - I$ and $g(y)$ as ρ changes. The top figure is for $\gamma = 5$, and the bottom figure is for $\gamma = 1$. Other parameter values are set as $\beta = r = 2\%$, $\alpha_y = 0.1$, $\sigma_y = 20\%$, $\eta = 0.2$, and $I = 10$.