Linearity-Inducing Processes: A Modelling Tool Yielding Closed Forms for Asset Prices

Xavier Gabaix
MIT, Princeton and NBER

October 30, 2006*
Preliminary and Incomplete – Comments Welcome.

Abstract

This methodological paper proposes a new class of stochastic processes with appealing properties for theoretical or empirical work in finance and macroeconomics, the "linearity-inducing" class. Its key property is that it yields simple exact closed-form expressions for stocks and bonds, with an arbitrary number of factors, feats that are not possible with the hitherto available modelling methods. It operates in discrete and continuous time. It has a number of economic modeling applications. These include macroeconomic situations with changing trend growth rates, asset pricing with time-varying risk premia or time-varying dividend growth rates, and yield curve analysis that allows flexibility and transparency. Many research questions may be addressed more simply and in closed form by using the linearity-inducing class. (JEL: G12, G13)

Keywords: Modified Gordon growth model, Stochastic Discount Factor, Linearity-inducing process, Affine models, Long term risk, Growth rate risk, Interest rate processes, Yield curve, Bond pricing.

---

[*] xgabaix@princeton.edu. Oleg Rytchkov provided very good research assistance. For helpful conversations, I thank Marco Avellaneda, John Campbell, Peter Carr, Sylvain Champonnois, John Cox, Greg Duffee, Darrell Duffie, Pierre Collin-Dufresne, Alex Edmans, Sydney Ludvigson, Lasse Pedersen, José Scheinkman, and workshop participants at MIT, NYU, and Princeton. I thank the NSF for support.

1 Introduction

This methodological paper proposes a new class of stochastic processes that has a number of attractive properties for economics and finance, the "linearity-inducing" (LI) process. It is generates closed-form solutions for the prices of stocks and bonds. It is simple and flexible, applies to an arbitrary number of factors with a rich correlation structure, and works in discrete or continuous time. These features make it an easy-to-use new tool for pure and applied financial modelling.

LI processes have the drift of an Ornstein-Uhlenbenck processes, with an additional, especially tailored term. In many applications, this extra term does not change the drift much quantitatively, so that LI processes are in effect fairly close to an Ornstein-Uhlenbeck process. However, adding the extra term makes tractable the expressions of expected values of stocks and bonds. The expressions are linear in the factors – hence the name "linearity-inducing" processes.

We obtain closed forms for the price of stocks with a time-varying growth rate, and the price of perpetuities. Such closed forms for stocks is not available with the existing processes, such as those of Ornstein-Uhlenbeck / Vasicek (1977), Cox, Ingersoll, Ross (1985), the Heath, Jarrow, Morton (1992), or models in the affine class (Duffie and Kan 1996). In fact, the only known prior example of a closed form for a stock with a stationary, stochastic growth rate or equity premium in continuous time, is Bhattacharya (1978) and, in another form, Menzly, Santos and Veronesi (2004), and this process turns out to be to be a particular case of the LI class (see Example 7). Hence, all known forms of closed forms for stocks (with a stationary stochastic structure) appear to belong to the LI class.

Linearity-inducing processes are meant to be a practical tool for several topics in economics. They are likely to be useful in: (i) macroeconomics, for models in which GDP growth rate is time-varying; (ii) asset pricing, for models where the equity premium is time-varying, and models where the trend growth rate of dividends is varying; and (iii) fixed-income analysis. Also, the process may be useful for thinking about other situations where the uncertainty in the discount factor matters, e.g., in environmental economics.

Several literatures motivate the need for a tool such as the LI process. Many recent studies investigates the importance of long-term risk for asset pricing and macroeconomics, e.g., Bansal and Yaron (2004), Barro (2006), Bekaert et al. (2005), Croce, Lettau and Ludivgson (2006), Gabaix and Laibson (2002), Hansen Heaton and Li (2005), Hansen and Scheinkman (2006), Julliard and Parker (2004), Lettau and Wachter (2007), Parker (2001). The LI process offers a way to model long-term risk, while keeping a closed form for stock prices. In addition, there is debate about the existence and mechanism of the time-varying expected stock market returns, e.g., Campbell and Shiller (1988), Cochrane (2006), and many others. Because of the lack of closed forms, the literature relies on simulations and approximations. The LI process offers closed forms for stocks with time-varying equity premium, which is useful for thinking about those issues.

Finally, we contribute to the vast literature on interest rate processes, by presenting a new, flexible process. It is as flexible as the affine class of Duffie and Kan (1996), which includes the Vasicek (1978) and the Cox, Ingersoll, Ross (1985) process as special cases. Section 5.3 develops the link between the LI class and the affine class.

This paper stipulates a process for finding desirable properties for the pricing kernel, and in this follows a productive literature represented by, e.g., Abel (2007), Campbell and Cochrane (1999), Cox, Ingersoll, Ross (1985), Pastor and Veronesi (2005), and, particularly, Menzly, Santos and Veronesi (2004).

Section 2 presents some simple examples of LI processes. Section 3 presents the discrete-time version of the process. Section 4 presents the basic one-factor process in continuous time. Section 5 presents the multifactor version of the LI process, which is useful for talking about processes with fast and slowly mean-reverting components, or interest rate processes with short and long rates. Section 6 shows some extensions, one to option pricing, one to time-dependent coefficients. Section 7, which is more technical, studies the range of admissible initial conditions. Section 8 concludes.

Most proofs are in the Appendix, except very short ones, and those that have a special interest to illustrate how to use LI processes.

2 A simple introduction to linearity-inducing processes

2.1 A basic example: the generalized Gordon formula

To get the flavor for linearity-inducing (LI) processes, we start with a simple example. Consider a stock with dividend $D_t = D_0 \exp\left(\int_0^t g_u du\right)$. g_t is the (stochastic) growth rate, and can be decomposed $g_t = g_* + \gamma_t$, where g_* is a trend growth rate, and γ_t a fluctuation around the trend. The discount rate is r, and the value of a stock at time t is, assuming $R = r - g_* > 0$,

$$P_t = E_t\left[\int_t^\infty \exp\left(-r\left(s-t\right)\right) D_s ds\right] = E_t\left[\int_t^\infty \exp\left(-\int_t^s \left(r - g_* - \gamma_u\right) du\right) ds\right] D_t$$

so that the price-dividend ratio is:

$$P_t/D_t = E_t\left[\int_t^\infty \exp\left(-R\left(s-t\right) + \int_t^s \gamma_u du\right) ds\right]. \tag{1}$$

This paper proposes a process for γ_t that yields a closed-form for (1). Before doing this, it is useful to examine the most natural process, which is to take γ_t to be an Ornstein-Uhlenbeck,

$$d\gamma_t = -\phi\gamma_t dt + \sigma dB_t.$$

Calculating (1) yields:

$$P_t/D_t = \int_0^\infty \exp\left[-RT + \frac{1-e^{-\phi T}}{\phi}\gamma_t + \frac{\sigma^2}{2\phi^3}\left(\phi T + 2e^{-\phi T} - \frac{e^{-2\phi T}+3}{2}\right)\right] dT \tag{2}$$

which is complicated and has no known closed-form expression. Likewise, a Cox, Ingersoll Ross (1985) process does not yield a closed form for the stock price.

The key idea of this paper springs from remarking that the a slight modification of the growth process has makes prices completely tractable. Consider the growth process:

$$d\gamma_t = -\phi\gamma_t dt - \gamma_t^2 dt + \sigma\left(\gamma_t\right) dB_t$$

where $\sigma\left(\gamma_t\right)$ is an essentially arbitrary function (more on this later), but the coefficient on γ_t^2 must be -1. Then, the following result holds.

Example 1 *(Generalized Gordon growth formula with LI stochastic trend growth rate) Consider a stock with dividend growth rate* $g_t = g_* + \gamma_t$, *with*

$$d\gamma_t = -\left(\phi\gamma_t + \gamma_t^2\right)dt + \sigma\left(\gamma_t\right)dB_t, \tag{3}$$

and price $P_t = E_t\left[\int_t^\infty \exp\left(-rt\right)D_s ds\right]$. *If the process is defined in* $[t, \infty)$, *the price-dividend ratio,* P_t/D_t *is:*

$$P_t/D_t = \frac{1}{r-g_*}\left(1 + \frac{\gamma_t}{r-g_*+\phi}\right). \tag{4}$$

The above example exhibits general traits of LI processes.[1]

1. As in (4), the price of assets are linear (affine) in the state variable – here, γ_t, which motivates the name "linearity-inducing" process for (3).

2. Surprisingly perhaps, the volatility term $\sigma\left(\gamma_t\right)$ does not appear in the final expression of the stock price. $\sigma\left(\gamma_t\right)$ can by multiplied by any number, the stock price does not change This gives much modelling flexibility.

3. We need an extra term in the drift process, here $-\gamma_t^2 dt$. In many applications, the term is likely to be small quantitatively. For instance, if we think that the deviation from the $|\gamma_t|$ is less than 5%/year typically (which is plausible for the predictable deviation from the trend growth rate, or the trend interest rate), then the extra drift term *is* less than $(5\%)^2 = 0.25\%$ per year. Hence, often, the extra drift term will not materially change the importance quantitative properties of the process. However, it confers a great tractability to asset prices.

4. Some care must be taken to make the process defined in $[t, \infty)$. This will be developed later in the paper, and is illustrated in Figure 1. In the context at hand, a sufficient condition is that $\sigma(\gamma)$ vanishes in a right neighborhood of $\gamma = -\phi$, and that the initial value of γ_t is above $-\phi$. This is analogous to the fact that the volatility must go to 0 as the interest rate goes to 0 in the Cox, Ingersoll and Ross (1985) process.

Insert Figure 1 about here

The economic interpretation of (4) is the following. When the deviation of the growth rate from its trend ($\gamma_t = g_t - g_*$) is 0, then $P_t/D_t = 1/\left(r-g_*\right)$, which is the traditional Gordon formula. When the growth rate is above trend ($\gamma_t > 0$), the P/D ratio is higher, as future dividends have superior growth. This initial superior growth γ_t decays at rate ϕ, and is discounted at rate $r - g_*$, so that its total duration is $1/\left(r-g_*+\phi\right)$. So the cumulative impact of the superior growth is hence the $\gamma_t/\left(r-g_*+\phi\right)$.

Let us now see why the price is a linear function of the initial growth rate.

---

[1] The result in Example 1 appear new to the literature. The Fisher-Wright process (e.g., Karlin and Taylor 1982) does contain a quadratic term, but it has not been applied to the pricing bonds or stocks. Also, it is more special than the LI class, because it imposes a specific functional form on the variance. Driessen, Maenhout and Vilkov (2005) apply the Fisher-Wright process to options. Other papers introduce different quadratic terms in stochastic process, for instance Ahn et al. (2002), Constantidines (1992), Lonstaff (1989), but they do not take the form of this paper.

A heuristic proof The proof of the result will be made fully rigorous in the rest of the paper, but a simple "plug and verify" derivation is instructive. Call the price-dividend ratio $V_t = P_t/D_t = E_t \int_t^\infty \exp\left(-\int_t^s (R-\gamma_u)\,du\right) ds$, with $R = r - g_*$. It is analogue to the price of a bond that gives 1 in every second, with an instantaneous interest rate of $R - \gamma_u$. Hence, the arbitrage equation for V_t is:

$$0 = 1 - (R-\gamma) V_t + E_t [dV_t]/dt.$$

As γ_t is the only state variable as far as V_t is concerned, we seek a solution of the form $V_t = V(\gamma_t)$. Call the drift of γ, where the drift of γ is

$$\mu(\gamma) = -\phi\gamma - \gamma^2$$

Ito's lemma gives: $E_t [dV_t]/dt = \mu(\gamma) V'(\gamma) + \frac{\sigma^2(\gamma)}{2} V''(\gamma)$, and the arbitrage equation is the classic equation:

$$0 = 1 - (R-\gamma) V(\gamma) + \mu(\gamma) V'(\gamma) + \frac{\sigma^2(\gamma)}{2} V''(\gamma) \tag{5}$$

We look for a solution affine in γ: $V(\gamma) = A + B\gamma$. This functional form implies $V''(\gamma) = 0$, so that, if the solution is correct, the $\sigma^2(\gamma)$ term will not matter. That fact explains why there are no σ terms in the final expression (4).

Substituting $V(\gamma) = A + B\gamma$ into (5), yields:

$$\begin{aligned} 0 &= 1 - (R-\gamma)(A+B\gamma) + \left(-\phi\gamma - \gamma^2\right) B + \frac{\sigma^2(\gamma)}{2} \cdot 0 \\ &= 1 - RA + \gamma(A - RB - \phi B) + \gamma^2 (B - B) \end{aligned} \tag{6}$$

The key simplification is that the terms in γ^2 cancel out – this is very the LI tern γ_t^2 matters. To solve the last equation, we just set to 0 the constant and the γ term, which gives $A = 1/R$, and $B = A/(R+\phi)$, which gives.

$$V(\gamma) = \frac{1}{R}\left(1 + \frac{\gamma}{R+\phi}\right)$$

which is the announced result, as with $R = r - g_*$.

If the term γ^2 had been absent of the drift (as in an Ornstein-Uhlenbeck process), or been present with a coefficient different from -1, the cancellation of the γ^2 in (6) would not have occurred. □

Insert Figure 2 about here

There is an intuitive reason why the price can be linear in the initial growth rate g_0. The price, a sum of $\exp\left(\int_0^T g_t dt\right)$, is a convex function of future growth rates g_t (see Pastor and Veronesi (2003) for a use of this fact). But, for instance in the deterministic version of the process, future growth rates are a *concave* function of the initial growth rate, $E_0 [g_t \mid g_0]$ is concave in g_0.[2] Hence the price is a composition of a convex function (namely, $\exp\left(\int_0^T g_t dt\right)$), composed with a concave function,

---

[2] If the process is deterministic, then $\gamma_t = e^{-\phi t}\gamma_0 / \left(1 + \gamma_0\left(1 - e^{-\phi t}\right)/\phi\right)$, a concave function. This can be shown directly, or by Proposition 3.

(namely, $g_t(g_0)$) the initial growth rate. Hence, its concavity is undertermined. For the LI process, the price is precisely a linear function of the initial growth rate.

The reader may wonder if this example is not arbitrary. Actually, it is not, and it derives from the principles of LI processes exposed below.

The next example shows an example with several factors.

2.2 A second example: A price-dividend ratio with time-varying growth rate and risk-premium

The LI processes generalize easily to several factors. Consider a world where stochastic discount factor M_t and the dividend process D_t follow

$$\begin{aligned} dM_t/M_t &= -rdt - \frac{\pi_t}{\sigma}dz_t \\ dD_t/D_t &= g_t dt + \sigma dz_t \end{aligned}$$

The price of the stock is $P_t = E_t\left[\int_t^\infty M_s D_s ds\right]/M_t$.

We assume that π_t and g_t follow the following LI process, best expressed in terms of their deviation from trend, $\widehat{\pi}_t = \pi_t - \pi_*, \widehat{g}_t = g_t - g_*$,

$$\begin{aligned} d\widehat{g}_t &= -\phi_g \widehat{g}_t dt + \widehat{g}_t\left(\widehat{\pi}_t - \widehat{g}_t\right)dt + \sigma_\gamma\left(\widehat{g}_t, \widehat{\pi}_t\right)\cdot dB_t \\ d\widehat{\pi}_t &= -\phi_\pi \widehat{\pi}_t dt + \widehat{\pi}_t\left(\widehat{\pi}_t - \widehat{g}_t\right)dt + \sigma_\pi\left(\widehat{g}_t, \widehat{\pi}_t\right)\cdot dW_t \end{aligned}$$

where the (B_t, W_t) is a Gaussian process independent of z_t and suppose that $\sigma_\gamma, \sigma_\pi$ ensure that the process is defined in $[t, \infty)$. Again the process for γ_t is essentially an Ornstein-Uhlenbeck, with this time a coupling term $\gamma_t\left(\widehat{\pi}_t - \gamma_t\right)$ added.

Under those assumption, one can write:

$$P_t/D_t = E_t\left[\int_t^\infty \exp\left(-\int_t^s \left(r + \pi_s - g_s\right)du\right)ds\right]$$

so that the stock has a time-varying risk-premium π_t, and time-varying growth rate of dividends, g_t. The LI terms imply the following Proposition.

Example 2 *(Generalized Gordon formula, with stochastic trend in dividend growth, and stochastic equity premium) In the above setup, the price-dividend ratio is, with $R = r + \pi_* - g_*$,*

$$P_t/D_t = \frac{1}{R}\left(1 + \frac{g_t - g_*}{R + \phi_g} - \frac{\pi_t - \pi_*}{R + \phi_\pi}\right). \tag{7}$$

In this expression the price-dividend ratio varies because of a stochastic equity premium (π_t), and a stochastic dividend growth rate (g_t).

It is a good and simple exercise to derive the above formula directly, from the arbitrage equation $1 - \left(r + \pi_t - g_t\right)V_t + E\left[dV_t\right]/dt = 0$. Another proof is that it comes from Theorem 4 below.

Equation (7) nests the two main sources of variations (in expected dividends, and in expected returns), in a simple and natural way. The two sources enter linearly, weighted by their duration (e.g., $1/\left(R+\phi_\pi\right)$), which depends of the speed of mean-reversion of the each process (parametrized by ϕ_π, ϕ_g), and the effective discount rate, R. As in the previous example, the volatility terms do not enter in (7), and the price does not change if one changes the correlation between the instantaneous innovation in g_t and π_t.

2.3 Third example: a LI multifactor interest rate process

We now move from stocks to bonds.

Example 3 *(A multifactor bond model). Suppose that the interest is r_t with $r_t = r_* + \sum_{i=1}^n r_{it}$, where each factor r_{it} follows, for $t \in \mathbb{R}_+$*

$$E\left[dr_{it}\right]/dt = -\phi_i r_{it} + \left(r_t - r_*\right) r_{it}$$

The interpretation is that r_* is the central value of the interest rate, and the r_{it} are transitory fluctuations around it, reverting essentially at rate ϕ_i, but with a modulation $\left(r_t - r_*\right) r_{it}$. For instance, if $\phi_1 < ... < \phi_n$, as r_{1t} mean-reverts at the slowest rate, r_{1t} affects relatively more long-maturity bonds, and r_{nt} relatively more short-maturity bonds. The correlation between the r_{it} can be left arbitrary. Suppose the agents are risk neutral, so that the price a bond of maturity T is $Z_t\left(T\right) = E_t\left[\exp\left(-\int_t^{t+T} r_s ds\right)\right]$. The process is LI, and the expression for the bond is:

$$Z_t\left(T\right) = e^{-r_* T}\left(1 - \sum_{i=1}^n \frac{1-e^{-\phi_i T}}{\phi_i} r_{it}\right) \tag{8}$$

Here, bond prices are linear in the factors. Later in the paper, we generalize this example to include risk premia.

The next section starts the more systematic presentation of LI processes.

3 Linearity-inducing processes in discrete time

This section studies the discrete-time version of the LI process. We want to price an asset with dividend D_t, given a discount factor M_t. The price at time t of a claim yielding a stochastic dividend D_s at date $S \geq t$ is:[3]

$$P_t = E\left[\sum_{T=0}^{\infty} M_{t+T} D_{t+T}\right]/M_t. \tag{9}$$

---

[3] Some readers may not be familiar with the stochastic discount factor. The simplest example is $M_t = (1+r)^{-t}$, if the interest rate is constant. If the interest rate r_s is deterministic but not constant, $M_t = \prod_{s=0}^{t}\left(1+r_s\right)^{-1}$. If, in Lucas economy, a representative consumer with utility $\sum_t \delta^t U\left(C_t\right)$ prices assets, then $M_t = \delta^t U'\left(C_t\right)$. Absence of arbitrage guaranties that the price is a linear functional of future dividends, and under weak technical conditions this leads to the existence of factors M_{t+T} such that (9) holds.

For instance, the price of a zero coupon bond of maturity T is, with $D_t = 1$,

$$Z_t(T) = E_t[M_{t+T}D_{t+T}] / (M_t D_t). \tag{10}$$

For instance, the gross short term interest from t to $t+1$ is $M_t/E_t[M_{t+1}]$. We will also calculate the price-dividend of a stock, $V_t = P_t/D_t$:

$$V_t = E_t \sum_{T=0}^{\infty} \frac{M_{t+T}D_{t+T}}{M_t D_t} = \sum_{T=0}^{\infty} Z_t(T) \tag{11}$$

3.1 Definition and main properties

The state vector is $X_t \in \mathbb{R}^n$, and we define m_t to be the growth of the dividend-augmented stochastic discount factor[4]

$$m_{t+1} = \frac{D_{t+1}M_{t+1}}{D_t M_t} \tag{12}$$

Definition 1 *The process $(m_t, X_t)_{t=0,1,2,...}$, with $m_t \in \mathbb{R}$ and $X_t \in \mathbb{R}^n$, is a LI process if the following relations hold, for all integers t*

$$\begin{aligned} E_t[m_{t+1}] &= \alpha + \delta' X_t & (13) \\ E_t[m_{t+1}X_{t+1}] &= \gamma + \Gamma X_t & (14) \end{aligned}$$

with $\alpha \in \mathbb{R}, \beta, \delta \in \mathbb{R}^n, \Gamma \in \mathbb{R}^{n^2}$.

Another way to see things is useful. Writing $M_t D_t = M_0 D_0 m_1 ... m_t$ the dividend-augmented SDF, and the process with values in $\mathbb{R}^{n+1}$

$$Y_t := \begin{pmatrix} M_t D_t \\ M_t D_t X_t \end{pmatrix}$$

and the matrix

$$\Omega = \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix} \tag{15}$$

conditions (13)-(14) can be written:

$$E_t[Y_{t+1}] = \Omega Y_t. \tag{16}$$

Hence, there is a $(n+1)$ dimensional process Y_t, and a vector $\nu' = (1, 0, ..., 0)$, such that (16) holds, and

$$M_t = \nu' Y_t \tag{17}$$

The (dividend-augmented) stochastic discount factor of a LI process is simply the projection of an autoregressive process, Y_t.

It is not difficult to write economic models satisfying conditions (13)-(14), see the examples below.

The basic properties are the following.

---

[4] The reader interested in bonds can think $D_t = 1$ throughout, which means that m_t is the stochastic discount factor. However, having applications in mind, it is easier to include a D_t term from the outset.

Theorem 1 *(Price of a zero-coupon bond in the multifactor, discrete time case) The price-dividend (10) of a zero-coupon equity or bond of maturity T is, with I_n the identity matrix of dimension n*

$$Z_t(T) = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}^T \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \tag{18}$$

When $\gamma = 0$, it can be expressed:

$$Z_t(T) = \alpha^T + \delta' \frac{\alpha^T I_n - \Gamma^T}{\alpha I_n - \Gamma} X_t \tag{19}$$

For instance, the above Theorem can price bonds, with n factors, in closed form.

The second main result is mostly useful for stocks.

Theorem 2 *Suppose that the process is defined from t on, that all eigenvalues of Ω have a modulus less than 1 (finiteness of the price). Then, the price-dividend ratio of the stock 11) is:*

$$\begin{aligned} P_t/D_t &= \frac{1}{1-\alpha-\delta'(I_n-\Gamma)^{-1}\gamma}\left(1+\delta'(I_n-\Gamma)^{-1}X_t\right) && (20) \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \left(I_{n+1} - \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}\right)^{-1} \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix}. && (21) \end{aligned}$$

Theorem 2 allows to generate stock prices with an arbitrary number of factors, including time-varying growth rate, and risk premia.

3.2 Some examples

Example 4 *A Gordon growth formula with time-varying dividend growth.*

Suppose that the interest rate is constant at r, and the growth rate of dividend is:

$$\begin{aligned} \frac{D_{t+1}}{D_t} &= (1+g_*)(1+x_t) && (22) \\ E_t[x_{t+1}] &= \frac{\rho x_t}{1+x_t} && (23) \end{aligned}$$

x_t is the deviation from the trend growth rate. If x_t was an AR(1), it would follow $E_t[x_{t+1}] = \rho x_t$. Instead, the process is slightly modified, to (23), to make the process linearity-inducing. Indeed, with $X_t = x_t/(1+x_t)$, the above is the LI process. The price is $P_t = E_t \sum_{s=0}^{\infty} D_{t+s}/(1+r)^s$, and its calculation yields:

$$P_t/D_t = \frac{1+r}{r-g_*}\left(1 + \frac{(1+g_*)\rho}{1+r-(1+g_*)\rho}\frac{x_t}{1+x_t}\right) \tag{24}$$

It is elementary to derive Eq. (24) directly, using $V_t = 1 + E_t[m_{t+1}V_{t+1}]$, and looking for a solution of the type $V_t = A + Bx_t/(1+x_t)$. We show here how it comes from the general LI structure – in fact, this example, and postulate (23), is dictated by the general principles of the LI processes.

Derivation of (24). Call $\alpha = \frac{1+g_t}{1+r}$, and define $X_t = x_t/\left(1+x_t\right)$, $m_t = \frac{1}{(1+r)}\frac{D_{t+1}}{D_t} = \alpha\left(1+x_t\right)$. We have: $E_t\left[m_{t+1}\right] = \alpha + \alpha X_t$, and

$$E_t\left[m_{t+1}X_{t+1}\right] = E_t\left[\alpha\left(1+x_{t+1}\right)\cdot\frac{x_{t+1}}{\left(1+x_{t+1}\right)}\right] = \alpha E_t\left[x_{t+1}\right] = \alpha\rho X_t$$

So we have $E_t\left[m_{t+1}\overline{X}_{t+1}\right] = \Omega\overline{X}_t$, with $\Omega = \begin{pmatrix}\alpha & \alpha\rho \\ 0 & \alpha\rho\end{pmatrix} =: \begin{pmatrix}\alpha & \delta' \\ \gamma & \Gamma\end{pmatrix}$. Hence, we apply Theorem 2, with a dimension $n=1$, $\gamma=0$, $\delta=\Gamma=\alpha\rho$. We get:

$$P_t/D_t = \frac{1}{1-\alpha-\delta'\left(I_n-\Gamma\right)^{-1}\gamma}\left(1+\delta'\left(I_n-\Gamma\right)^{-1}X_t\right) = \frac{1}{1-\alpha}\left(1+\frac{\alpha\rho}{1-\alpha\rho}\frac{x_t}{1+x_t}\right).$$

Formula (24) is the discrete-time analogue of (4), with very small r and g_*, and the substitutions $\rho = 1-\phi$, for a small ϕ.

Example 5 *Flexible LI parametrization of the state variables, and the stochastic discount factor*

Take an $n-$dimensional process x_t, such that

$$E_t\left[x_{t+1}\right] = \gamma + \frac{\Gamma x_t}{\alpha+\beta'\left(x_t-\gamma\right)} \tag{25}$$

and define:

$$\begin{aligned} m_t &= \alpha+\beta'\left(x_t-\gamma\right)+\eta_{t+1} && (26)\\ X_t &= \frac{x_t}{\alpha+\beta'\left(x_t-\gamma\right)} && (27)\end{aligned}$$

with $E_t\left[\eta_{t+1}\right]=0$ and $E_t\left[X_{t+1}\eta_{t+1}\right]=0$. Then, m_t and X_t satisfy (13)-(14), with $\delta' = \beta'\Gamma$.

The above equations are give the LI counterpart of the popular "affine" parametrization, $m_{t+1} = \exp\left(A+B'X_t\right)$, $X_{t+1} = \Gamma X_t + b + \varepsilon_{t+1}$, with ε_{t+1} Gaussian. It is as flexible.

To interpret (25), consider the case $\gamma=0$. Eq. (25) expresses that, when x_t is small,

$$E_t\left[x_{t+1}\right] = \frac{\Gamma x_t}{\alpha+\beta' x_t} \sim \frac{\Gamma}{\alpha}x_t$$

which means that x_t follows approximately at AR(1). The corrective $1+\beta'/\alpha\cdot x_t$ in the denominator is often small in practice, but ensures that the process is LI.

Section 7 provides conditions to ensure $m_{t+1}>0$ for all times.

4 The one-factor linearity-inducing process in continuous time

Before presenting the LI process in n dimensions, it may be useful to present it in one dimension. We fix a probability space $\left(\Omega^P,\mathcal{F},P\right)$ and an information filtration $\mathcal{F}_t$ satisfying the usual technical conditions (see, for example, Karatzas and Shreve 1991). We start from a simple Ornstein-Uhlenbeck process for the short-term interest rate r_t, but add the extra LI term:

$$dr_t = -\phi \left(r_t - r_*\right) dt + \left(r_t - r_*\right)^2 dt + dN_t, \tag{28}$$

where N_t is a martingale, and $\phi > 0$, and $r_t \leq \overline{r} \in (r_*, r_* + \phi)$.

Term $-\phi \left(r_t - r_*\right) dt$ is the standard term which leads to mean reversion to the long run value r_*, with a speed ϕ. The second term, $\left(r_t - r_*\right)^2$, represents a small departure from a classical Ornstein-Uhlenbeck process. When r_t is not too far from r_*, the drift is little changed from the standard $-\phi \left(r - r_*\right) dt$. However, the correction $\left(r_t - r_*\right)^2 dt$ substantially improves the tractability of the process. We require the leading terms to be $\left(r_t - r_*\right)^2$ in the definition (28) of the LI process, not $q \left(r_t - r_*\right)^2$ for a $q \neq 1$.

We assume that the martingale $\{N_t, t \geq 0\}$ has properties that ensure the interest rate remains below some upper bound $\overline{r} \in (r_*, r_* + \phi)$, and thus does not explode. One example is $dN_t = \sigma \left(r_t\right) dz_t$ where z_t is a Brownian process and $\sigma \left(r\right) \sim k \left(\overline{r} - r\right)^{\kappa}$, for r in a left neighborhood of $\overline{r}$, $\kappa > 1/2$ and $k > 0$. Given the drift is negative around $\overline{r}$, that will ensure that $\overline{r}$ is a natural boundary, and $\{\forall t, r_t \leq \overline{r}\}$ almost surely, as detailed in Appendix A. We also have the option to set a lower bound $\underline{r}$ to the interest rate, by adding an additional restriction on dN_t. For instance, if one wishes to ensure $r_t \geq \underline{r}$ for some $\underline{r} < r_*$, we could have $dN_t = \sigma \left(r_t, t\right) dz_t$, with $\sigma_t \sim k' \left(r - \underline{r}\right)^{\kappa'}$, $\kappa' > 1/2$ for r in a right neighborhood of 0, and $k' > 0$.

Appendix A contains a formal statement and justification of these assumptions, and further purely technical assumptions. Given these assumptions, and an initial value r_0, the stochastic differential equation (28) has a unique strong solution for r_t, defined for all times $t \geq 0$.

The following Proposition gives a basic result. Its proof is in Appendix B.

Proposition 1 *The price at time t of a zero-coupon bond of maturity T, $Z_t \left(T\right) = E_t \left[\exp \left(-\int_t^{t+T} r_s ds\right)\right]$, is:*

$$Z_t \left(T\right) = e^{-r_* T} \left(1 + \frac{e^{-\phi T} - 1}{\phi} r_t\right). \tag{29}$$

In this expression (29), we find the same traits as in the generalized Gordon formula (4). The bond price is linear in the current rate r_t, hence the "linearity-inducing" moniker.

Also, bond price does not depend on the specific of the noise dN_t and thus the volatility of the process. If $dN_t = \sigma \left(r_t\right) dz_t$, the bond price does not depend on σ. The intuition is as follows. Consider the case where the process is deterministic, i.e., $dN_t \equiv 0$ for all t. Proposition 1 still applies, and we rewrite Eq. 29 as: $Z \left(T, r\right) = A + Br$, where A and B are constants that depend on T but not on r. Suppose that at time t the interest rate is r_t, and apply a shock to the interest rate at time $t + dt$ so that $r_{t+dt} = r_t + u_t$, where u_t is a mean-zero random variable. After $t + dt$, the process remains deterministic, and evolves according to (28) with $dN_t = 0$. Given an initial value $r_t + u_t$ of the interest rate, the bond price at $t + dt$ is $Z \left(T, r_t + u_t\right) = Z \left(T, r_t\right) + Bu_t$, because Z is affine in r. Hence the value at t is $E \left[Z \left(T, r_t + u_t\right)\right] = Z \left(T, r_t\right)$. The expected value of the impact of u_t is 0, because the bond price is linear in u_t. A zero mean addition of volatility does not affect the bond price. The crux is that, in the deterministic case ($\forall t, M_t = 0$), the bond price is an affine function of the interest rate: $Z \left(T, r_t\right) = A \left(T\right) + B \left(T\right) r_t$. By contrast, intuition based on the expression $E_t \left[\exp \left(-\int_t^{t+T} r_s ds\right)\right]$ would suggest a convex function.

The independence of bond prices from volatility greatly simplifies the analysis. In particular, dN_t could have jumps, which model a decision by the central bank. One does not need to specify the volatility process to get the prices of bonds: only the drift part is necessary. This leaves a high margin of flexibility to calibrate volatility, for instance on interest rate derivatives, a topic we do not pursue here.

Perhaps surprisingly, we can impose a lower bound on the interest rate (for instance, make sure that it remains always positive) without affecting bond prices, provided the lower bound $\underline{r}$ is less than r_*. The intuition is the following. We can make $\underline{r}$ a lower bound for the process simply by setting the variance to be 0 below $\underline{r}$. Since variance does not affect bond prices, the new lower bound does not affect bond prices.

Existing models, such as Vasicek and Cox-Ingersoll-Ross, are able to generate closed-form solutions for zero-coupon bonds. However, the LI process introduced by this paper can also generate closed-form solutions for perpetuities.

Proposition 2 *The price of a perpetuity is*

$$V_t = E_t \left[\int_0^\infty e^{-\int_t^{t+T} r_s ds} dT \right] = \frac{1}{r_*} \left(1 + \frac{r_t - r_*}{r_* + \phi} \right). \tag{30}$$

Proof. This follows directly from Eq. 29 and $V_t = \int_0^\infty Z_t(T)\, dT$. ■

We now analyze the dynamics of the process in some further detail. If we start from $r_t = r_*$, then $Z_t(T) = e^{-r_* T}$, which is the value the bond would have if r_t remained pinned at r_*. However, r_t is stochastic. Eq.29 shows that $Z_t(T) \sim (1 - r_t/\phi)\, e^{-r_* T}$ for $T \to \infty$. Hence r_* is the long run interest rate, $r_* = -\lim_{T\to\infty} Z'(T)/Z(T)$. Given (28), $E[dr_t] > 0$ if r is below r_*, and $E[dr_t] < 0$ if r is above r_*. Hence, r is mean-reverting towards r_*.

The process is defined only for $r_t \leq \overline{r} \leq r_* + \phi$. If $r_t > \overline{r}$, the process may explode in finite time. Indeed, when the process is deterministic ($M = 0$) and $r_t > \overline{r}$, then the process explodes in finite time, i.e., there is $t_1 > t$ such that $\lim_{s\to t_1} r_s = \infty$. The assumption $r_t < r_* + \phi$ is needed in the proof to ensure that the process is defined at times in $[t, t+T]$. This is guaranteed under the conditions of Appendix A.

The empirical relevance of the extra quadratic term is unclear. It could be, for instance, that the quadratic term exists under the risk-neutral probability, but not in the physical probability. Also, of course all processes are an approximation of reality, and in many cases, for instance for options, the major counterfactual assumption is probably to assume Gaussian innovations rather than fat-tailed ones (see Gabaix et al. 2003, 2006). LI processes allow for jumps and fat-tailed innovations, as the shape of the innovations (the dN_t term) does not affect the bond and stock prices – though, it would affect the option prices.

5 The multifactor linearity-inducing process in continuous time

Several factors are needed to capture the dynamics of stocks (Campbell and Shiller 1988, Fama and French 1996) and bonds (Litterman and Scheinkman 1991). Accordingly, we study the multifactor version of the LI process.

The discount factor is M_t. For applications, we will express the results in terms of a dividend-augmented SDF, $M_t D_t$. Often, it is better to imagine $D_t \equiv 1$.

5.1 Definition and main properties

The definition in continuous time is the limit of the definition in discrete time.

Definition 2 *The process* $(M_tD_t, X_t)_{t\in\mathbb{R}_+}$, *with* $M_tD_t \in \mathbb{R}$ *and* $X_t \in \mathbb{R}^n$, *is a LI process if the following relations hold, for all* $t \geq 0$,

$$\begin{aligned} E\left[\frac{d(M_tD_t)}{M_tD_t}\right]/dt &= -a-\beta' X_t && (31)\\ E\left[\frac{d(M_tD_tX_t)}{M_tD_t}\right]/dt &= b-(\Phi+aI_n)X_t && (32)\end{aligned}$$

with $a \in \mathbb{R}, b, \beta \in \mathbb{R}^n, \Phi \in \mathbb{R}^{n^2}$, *and* I_n *the identity matrix of dimension* $n \times n$.

For instance, in the case $D_t = 1$ and $dM_t/M_t = -\left(a+\beta' X_t\right)dt$, we have

$$dX_t = b - \Phi X_t dt + \left(\beta' X_t\right) X_t dt + dN_t \tag{33}$$

with $N_t \in \mathbb{R}^n$ is a martingale. So $E_t[dN_t] = 0$, but its component dN_{it}, dN_{jt} *can* be correlated. The simplest type of martingale is $dN_t = \sigma(X_t)\,dB_t$, for B_t a Brownian motion, but richer structures, e.g. with jumps, are allowed. As in the one-factor process, the volatility of dN_t must go to zero in some limit regions for the process to be well-defined. We defer this more technical issue until section 7.

With the notation $Y_t = \begin{pmatrix} M_tD_t \\ M_tD_tX_t \end{pmatrix} \in \mathbb{R}^{n+1}$, and

$$\omega = \begin{pmatrix} \alpha & \beta \\ -b & \Phi + aI_n \end{pmatrix} \tag{34}$$

one can write the conditions (31)-(32) more compactly as:

$$E_t[dY_t] = -\omega Y_t dt. \tag{35}$$

which is the analogue of (16). The above process leads to a discrete-time process with time increments Δt, with a matrix $\Omega = e^{-\omega\Delta t}$. When Δt is small, we have $\Omega = 1 - \omega\Delta t + O\left((\Delta t)^2\right)$.

In other terms, there is a autoregressive process Y_t in the background, following (35). The (dividend-augmented) stochastic is the one-dimensional projection of it. LI processes are tractable, because they are the one-dimensional projection of an AR(1) process.

The next Theorem prices claims of finite maturity.

Theorem 3 *(Bond prices). Given the LI process* (M_tD_t, X_t), *the price of a claim on a dividend of maturity* T, $P_t = E_t[M_{t+T}D_{t+T}]$, *satisfies:*

$$P_t/D_t = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \exp\left[-\begin{pmatrix} a & \beta' \\ -b & \Phi + aI_n \end{pmatrix} T\right] \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \tag{36}$$

an expression which, when $b = 0$, simplifies to:

$$P_t/D_t = e^{-aT}\left[1 + \beta' \frac{e^{-\Phi T} - I_n}{\Phi} X_t\right] \tag{37}$$

As an example, bond prices come from $D_t = 1$.

From this, we can now prove Theorem 4, which is probably the most useful of this section.

Theorem 4 *(Stock prices). Given the LI process $(M_t D_t, X_t)$, suppose that all eigenvalues of ω have positive real part (finite stock price). Then, the price/dividend ratio, $P_t/D_t = E_t\left[\int_t^\infty M_s D_s ds\right]/(M_t D_t)$, is:*

$$P_t/D_t = \frac{1 - \beta'(\Phi + aI_n)^{-1} X_t}{a + \beta'(\Phi + aI_n)^{-1} b} \tag{38}$$

Finally, the following Propositions show that one can price claims that have dividend a linear function of $D_t X_t$.

Proposition 3 *(Value of a single-maturity claim yielding $D_{t+T}X_{t+T}$). Given the LI process $(M_t D_t, X_t)$, the price of a claim on a dividend of maturity T, $P_t = E_t[M_{t+T}D_{t+T}X_{t+T}]/M_t$, satisfies:*

$$P_t/D_t = \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \exp\left[-\begin{pmatrix} a & \beta' \\ -b & \Phi + aI_n \end{pmatrix} T\right] \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \tag{39}$$

an expression which, when $b = 0$, simplifies to:

$$P_t/D_t = e^{-(aI_n + \Phi)T} X_t \tag{40}$$

Proposition 4 *(Value of a stock yielding $D_t X_t$) Under the conditions of Theorem 4, the price of a claim on a dividend of maturity T, $P_t = E_t\left[\int_t^\infty M_s D_s X_s ds\right]/(M_t D_t)$, satisfies:*

$$E_t\left[\int_t^\infty M_s D_s X_s ds\right]/(M_t D_t) = \frac{(\Phi + aI_n)^{-1}(b + aX_t)}{a + \beta'(\Phi + aI_n)^{-1} b}, \tag{41}$$

5.2 Some examples

We start with some stock-like examples.

Example 6 *Dividend growth rate as a sum of mean-reverting processes (e.g., a slow and a fast process).*

Suppose $D_T = D_0 \exp\left(\int_0^T g_t dt\right)$, with

$$\begin{aligned} g_t &= g_* + \sum_{i=1}^n X_{it} \\ E[dX_{it}]/dt &= -\phi_i X_{it} - (g_t - g_*) X_{it} \end{aligned}$$

The growth rate g_t is a steady state value g_*, plus the sum of mean-reverting processes X_{it}. Each X_{it} mean-reverts with speed λ_i, and also has the quadratic perturbation $(g_t - g_*) X_{it} dt$.

We apply the Theorem 3, with $\beta' = (1, ..., 1)$, $\Phi = Diag(\phi_1, ..., \phi_n)$. The price-dividend ratio is:

$$P_t/D_t = \frac{1}{r - g_*}\left(1 + \sum_{i=1}^{n} \frac{X_{it}}{r - g_* + \phi_i}\right). \tag{42}$$

Each component X_{it} perturbs the baseline Gordon expression $1/(r - g_*)$. The perturbation is X_{it}, times the duration of X_i, discounted at rate $r - g_*$, which is the term $1/(r - g_* + \phi_i)$.

Example 7 *The aggregate model of Menzly, Santos and Veronesi (2004), and the Bhattacharya (1978) mean-reverting process, belong to the linearity-inducing class.*

The following point is simple and formal. The Bhattacharya (1978) process is: $D_t = \overline{D} + \widehat{D}_t$, $d\widehat{D}_t = -\phi \widehat{D}_t dt + \sigma\left(\widehat{D}_t\right) dz_t$. It actually belongs to the LI class. Under another guise, it is used in the aggregate model of Menzly, Santos and Veronesi (2004), where S_t is their consumption-surplus ratio, which satisfies $E_t[dY_t] = k\left(\overline{Y} - Y_t\right) dt$, with $Y_t = 1/S_t$. The price-consumption ratio in their economy is $V_t = Y_t^{-1} E_t\left[\int_0^\infty e^{-\rho s} Y_{t+s}\right]$. In terms of the LI process, the state variable is $X_t := S_t$, and $M_t D_t := e^{-\rho t} Y_t$. We have $E_t[d(M_t D_t)/dt]/(M_t D_t) = -\rho - k + k\overline{Y} X_t$, and $E_t[d(M_t D_t X_t)/dt]/(M_t D_t) = -\rho X_t$, so that the MSV process satisfies the LI equations (31)-(32). Their Eq. 17 comes from Theorem 4 of the present article, with $\begin{pmatrix} a & \beta' \\ -b & \Phi + a I_n \end{pmatrix} = \begin{pmatrix} \rho + k & -k\overline{Y} \\ 0 & \rho \end{pmatrix}$. Hence, in retrospect, the Menzly, Santos and Veronesi (2004) process is tractable because it belongs to the LI class.

We next present some bond-like examples.

Example 8 *A multifactor bond model, with bond risk premia*

The following variant of Example 3 gives bond risk premia. Suppose $dM_t/M_t = -r_t dt + dN_t$, where N_t is a martingale, $r_t = r_* + \sum_{i=1}^{n} r_{it}$, and

$$E[dr_{it}] + \langle dr_{it}, dN_t \rangle = [-\phi_i r_{it} + (r_t - r_*) r_{it}] dt$$

Then the process is LI, and bond price is given by:[5]

$$Z_t(T) = e^{-r_* T}\left(1 - \sum_{i=1}^{n} \frac{1 - e^{-\phi_i T}}{\phi_i} r_{it}\right) \tag{43}$$

---

[5] As bond prices are independent of volatility, the process exhibits "unspanned volatility," a relevant feature of the data, as shown by Collin-Dufresne and Goldstein (2002). Of course, it could be the volatility depends on the factors directly, so that there would be a correlation between volatility and prices, but that would be an indirect correlation, rather than a direct one via the price formulas.

The risk-premium at t on the $T-$maturity zero coupon is, $\pi(T) := -\left\langle \frac{dZ_t(T)}{Z_t}, \frac{dM_t}{M_t} \right\rangle / dt$, is:

$$\pi(T) = \frac{\sum_{i=1}^{n} \frac{1-e^{-\phi_i T}}{\phi_i} \langle dr_{it}, dM_t/M_t \rangle}{1 - \sum_{i=1}^{n} \frac{1-e^{-\phi_i T}}{\phi_i} r_{it}} \tag{44}$$

With a parametrization for $\langle dr_{it}, dM_t/M_t \rangle = \langle dr_{it}, dN_t \rangle$, the above expression makes prediction for bond risk premia across maturities. It would be interesting to compare them with recent evidence, e.g. from Cochrane and Piazzesi (2005).

Example 9 *A model where the short rate and the long rate are factors*

r_t is the short term rate, call L_t is the long term rate, where, calling $r_t' = r_t - r_*, L_t' = L_t - r_*$

$$\begin{aligned} E_t\left[dr_t'\right] + \left\langle dr_t', dM_t/M_t \right\rangle &= \left[\phi\left(L_t' - r_t'\right) + \left(r_t'\right)^2\right] dt \\ E_t\left[dL_t'\right] + \left\langle dr_t', dM_t/M_t \right\rangle &= \left[-\psi L_t' + r_t' L_t'\right] dt \end{aligned}$$

Because of the first equation, r_t' is drawn to L_t'. The second equation expresses that L_t' goes to a steady-state value 0. The process is enriched by the terms $\left(r_t'\right)^2$ and $r_t' L_t'$, which make the process LI. We apply the Theorem 3, with $X_t' = \begin{pmatrix} r_t' \\ L_t' \end{pmatrix}, \beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \Phi = \begin{pmatrix} \phi & -\phi \\ 0 & \psi \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. We get:

$$Z_t(T) = e^{-r_* T}\left(1 - \frac{r_t'}{\phi} - \frac{L_t'}{\psi}\right) + e^{-(r_*+\psi)T} \cdot \frac{\phi}{\psi} \frac{L_t'}{\phi - \psi} + e^{-(r_*+\phi)T}\left(\frac{r_t'}{\phi} - \frac{L_t'}{\phi - \psi}\right)$$

and the perpetuity price is

$$V_t = \frac{1}{r_*} - \frac{r_t'}{r_*\left(r_* + \phi\right)} - \frac{\phi L_t'}{r_*\left(r_* + \phi\right)\left(r_* + \psi\right)}$$

Example 10 *A consistent model in the spirit of Brennan and Schwartz, where the factors are the short term rate, and the perpetuity rate*

Changing variables in the previous example, allows to answer an open question in the interest rate literature: is there a consistent model, whose factors are the short term rate and the perpetuity rate? Calling $c_t' = V_t - 1/r_*$, the deviation of the perpetuity price from its central value $1/r_*$, consider the following process:

$$\begin{aligned} E_t\left[dr_t'\right] + \left\langle dr_t', dM_t/M_t \right\rangle &= \left[-\left(\phi + \psi + r_*\right) r_t' - \left(\phi + r_*\right)\left(\psi + r_*\right) c_t' + r_t'^2\right] dt \\ E_t\left[dc_t'\right] + \left\langle dc_t', dM_t/M_t \right\rangle &= \left[r_t/r_* - r_* c_t' + c_t r_t'\right] dt \end{aligned}$$

and the short-term rate is $r_t = r_* + r_t'$, i.e. $E_t\left[dM_t/M_t\right] = -\left(r_* + r_t'\right) dt$. Again, in the simple case where $M_t = \exp\left(-\int_0^t r_s ds\right)$, then $\langle dr_t', dM_t/M_t \rangle = \langle dc_t', dM_t/M_t \rangle = 0$.

The price of a zero-coupon bond is:

$$\begin{aligned} Z_t(T) &= e^{-r_* T} + e^{-r_* T}\left(\frac{r_*}{\phi\psi} + \frac{(r_*+\phi)\,e^{-\phi T}}{\phi(\phi-\psi)} + \frac{(r_*+\psi)\,e^{-\psi T}}{\psi(\psi-\phi)}\right) r_t' + \\ &\quad + e^{-r_* T}\frac{r_*(r_*+\phi)(r_*+\psi)}{\phi\psi}\left(1 + \frac{\psi e^{-\phi T} - \phi e^{-\psi T}}{\phi-\psi}\right) c_t' \end{aligned}$$

while the price of the perpetuity is

$$V_t = 1/r_* + c_t'$$

In the above model, the factors are the short term rate $r_t = r_* + r_t'$, and the console price $V_t = 1/r_* + c_t'$, which is isomorphic to the console rate (also called the swap rate) $1/V_t$. Hence, we have a model that solves the open question that started with Brennan and Schwartz (1979): how to provide an arbitrage-free model interest rates, where the short rate, and the console rate, are factors. To best of our knowledge, this is the first model that answers that question.

Example 11 *r_t having a time-varying trend*

In the post-Volcker era, interest rates tended to have predictable trends of increase or increase, which may be captured by the following trend growth rate s_t of the interest rate:

$$\begin{aligned} E[dr_t]/dt &= s_t + (r_t - r_*)^2 \\ E[ds_t]/dt &= [-\lambda\mu(r_t - r_*) - (\lambda+\mu)s_t] + (r_t - r_*)s_t \end{aligned}$$

with $\lambda, \mu \geq 0$, and $\lambda + \mu > 0$. Economically, s_t is the predicted trend in interest rates, as per the first expression. s_t mean-reverts for two reasons: first, because of the $-\lambda\mu(r_t - r_*)$ term (s_t becomes negative if interest rates are too high); second, because of the $-(\lambda+\mu)s_t$ term.

We apply Theorem 3, with $X_t' = (r_t, s_t)$, $\beta' = (1, 0)$, $\Phi = \begin{pmatrix} 0 & -1 \\ \lambda\mu & \lambda+\mu \end{pmatrix}$. We get:

$$Z(T) = e^{-r_* T}\left[1 + \left(e^{-\lambda T} - 1\right)\frac{s_t + \mu(r_t - r_*)}{\lambda(\mu-\lambda)} - \left(e^{-\mu T} - 1\right)\frac{s_t + \lambda(r_t - r_*)}{\mu(\mu-\lambda)}\right]$$

and the perpetuity price is $P_t = \frac{1}{r_*} - \frac{s_t + (r_*+\lambda+\mu)(r_t - r_*)}{(r_*+\lambda)(r_*+\mu)}$.

Those examples show it is quite easy to get closed forms with easy to interpret processes.

5.3 Relation to the affine-yield class

The affine class (Duffie and Kan 1996; Duffie, Pan and Singleton 2000) is an important class, that contains the processes of Vasicek/Ornstein-Uhlenbeck (1977) and Cox, Ingersoll, Ross (1985). It is a workhorse of much theoretical and empirical work in asset pricing. It comprises processes of the type:

$$\begin{aligned} dX_t &= (b - \Phi X_t)\,dt + w_t dz_t \\ w_t w_t' &= \sigma^2\left(H_1' X_t + H_0\right) \end{aligned}$$

with $b, X_t \in \mathbb{R}^n$, $\Phi \in \mathbb{R}^{n\times n}$, $(H_0, H_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}, \sigma \in \mathbb{R}$, z_t is a $n-$dimensional Brownian motion. The interest rate is $r_t = r_* + \beta' (X_t - X_*)$, where $X_* = \Phi^{-1} b$, is assumed to exist.

Under mild technical conditions, bond prices have the expression:

$$Z_t^{\mathrm{Aff}}(T) = \exp\left(-r_* T + \Gamma(T)' (X_t - X_*) + \sigma^2 a(T)\right)$$

where $a(T)$ and $\Gamma(T)$ satisfy coupled ordinary differential equations, that typically need to be solved numerically. The situation is simpler if $H_1 = 0$. In that case, $\Gamma(T) = \gamma(T)$, with $\gamma(T)' = \beta' \left(e^{-\Phi T} - 1\right)/\Phi$. Then:

$$Z_t^{\mathrm{Aff}}(T) = \exp\left(-r_* T + \gamma(T)' (X_t - X_*) + \sigma^2 a(T)\right). \tag{45}$$

The express can be contrasted with the expression for the LI process (37):

$$Z_t^{\mathrm{LI}}(T) = e^{-r_* T}\left(1 + \gamma(T)' (X_t - X_*)\right). \tag{46}$$

If $\gamma(T)' X_t$ is small, the two expressions are the same, up to terms of second order in $\gamma(T)' X_t$, and second order in σ. Hence, a LI process is a good approximation if the underlying process is in fact affine, and vice-versa. In most cases, the two values are likely to be close, so that existing estimates of parameters in the affine class can be used to calibrate LI processes.[6]

What are the respective merits of the LI and affine classes? The main virtue of the LI process is that is yields explicit expressions for the prices of bonds and stocks. Another, and probably lesser, virtue is that it allows a free functional form for the innovation dN_t, which can include jumps and non-Gaussian behavior, and a free type of heteroskedascity.

In LI processes, the variance needs to go to 0 at the borders, may make the fit difficult, e.g. when pricing options. Section 6.3 indicates how to price options with the LI class. A potential drawback of pricing bonds with the LI process, is that, in the simplest version at least, bonds have no convexity in the LI framework. However, multifactor LI processes can have a flexible degree of convexity.

A nice property of affine process is that if M_t is in the affine class, and γ is a constant, then M_t^{γ} is in the affine class too. LI processes do not have that property. Otherwise, affine models are that they are well-understood, they have been estimated, and explicit economic models have been constructed with them. It would be very desirable to do the same for LI models.

Future research can assess their empirical merit of the LI class for bonds. However, LI process should be immediately useful where they are vasly more tractable than the affine class (Ang and Liu 2004) – in the pricing of stocks with time-varying growth rate and discount rate.

---

[6] That equivalence gives a useful way to calculate easily functionals of LI processes, that can be expressed as a linear combination of bonds. One first works with the affine process, setting volatility to 0, doing a first order Taylor expansion of terms in $(X_t - X_*)$. One gets an expression: $P_t^{\mathrm{Aff}} = a + b(X_t - X_*) + o(X_t - X_*) + o(\sigma^2)$, for some constant a, b. Then, one knows that for the corresponding LI process, the value of the asset is: $P_t^{\mathrm{LI}} = a + b(X_t - X_*)$, exactly.

6 Extensions

6.1 Processes with time-dependent coefficients

It is simple to extend the process to time-dependent coefficient. Suppose the process is:

$$\begin{aligned} E_t\left[\frac{d\left(M_t D_t\right)}{M_t D_t}\right] &= \left(-a\left(t\right)-\beta'\left(t\right)X_t\right)dt \\ E_t\left[\frac{d\left(M_t D_t X_t\right)}{M_t D_t}\right] &= \left(b\left(t\right)-\left(\Phi\left(t\right)+a\left(t\right)\right)X_t\right)dt \end{aligned}$$

With $Y_t=\left(M_t, M_t X_t\right)^{\top}$, this is $E\left[dY_t\right]/dt=-\omega\left(t\right)Y_t$, where $\omega\left(t\right)=\begin{pmatrix}\alpha\left(t\right) & \beta\left(t\right) \\ -b\left(t\right) & \Phi\left(t\right)+a\left(t\right)I_n\end{pmatrix}$. The solution is: $E_0\left[Y_T\right]=\exp\left(-\int_0^T\omega\left(t\right)dt\right)Y_0$. Hence, in the zero-coupon expressions, it is enough to replace ωT by $\int_0^T\omega\left(t\right)dt$. For instance, the equivalent of (37) is:

$$P_0/D_0=e^{-\int_0^T a(t)dt}\left[1+\left(\int_0^T\beta\left(t\right)dt\right)^{-1}\frac{e^{-\left(\int_0^T\Phi(t)dt\right)}-I_n}{\left(\int_0^T\Phi\left(t\right)dt\right)^{-1}}X_t\right].$$

6.2 Closedness under addition and multiplication

The product of two uncorrelated LI processes is LI. The same reasoning works for the product of two LI process. The product of two uncorrelated LI processes with respective dimensions d_1, d_2 (i.e., with d_1-1 and d_2-1 factor respectively) is LI, with dimension d_1d_2 (i.e., with d_1d_2-1 factors). The idea is simple, though it requires somewhat heavy notations.

We start in discrete time. Take two LI processes $\left(M_t^i, Y_t^i\right)$, and the product stochastic discount factor $m_t=m_t^1m_t^2$. Assume that, for any index i, j of the components, $E_t\left[Y_{t+1}^{1(i)}Y_{t+1}^{2(j)}\right]=E_t\left[Y_{t+1}^{1(i)}\right]E_t\left[Y_{t+1}^{2(j)}\right]$, a condition which is for instance verified if the processes are independent. Then, it is easy to verify that for any vector ψ^i, $E_t\left[\left(\psi^1Y_T^1\right)\left(\psi^2Y_T^2\right)\right]=E_t\left[\psi^2Y_T^2\right]E_t\left[\psi^2Y_T^2\right]$. In particular, $E_t\left[M_T^1M_T^2\right]=E_t\left[M_T^1\right]E_t\left[M_T^2\right]$

Then, $m_t=m_t^1m_t^2$ is also the SDF of a LI process.[7] The state vector is $\overline{Y}_t^1\otimes\overline{Y}_t^2$, i.e. the vector mades of the d_1d_2 components $\overline{Y}_t^{1(i)}\overline{Y}_t^{2(j)}$, $i=0...n_X, j=0...n_Y$. The corresponding Ω matrix is $\Omega=\Omega^1\otimes\Omega^2$. This comes simply from the fact that $E_t\left[m_{t+1}^Xm_{t+1}^Y\overline{X}_{t+1}\otimes\overline{Y}_{t+1}\right]=E_t\left[m_{t+1}^X\overline{X}_{t+1}\right]\otimes E_t\left[m_{t+1}^Y\overline{Y}_{t+1}\right]$.

In continous time, suppose $E\left[d\left(M_t^X\overline{X}_t\right)/dt\right]=-\omega^XM_t^X\overline{X}_tdt$, and $E\left[d\left(M_t^Y\overline{Y}_t\right)/dt\right]=-\omega^YM_t^Y\overline{Y}_tdt$. Then, $M_t^XM_t^Y$ is also a pricing kernel that comes from a LI process. The state vector is $\overline{X}_t\otimes\overline{Y}_t$ (which has dimension d_1d_2), and the ω matrix is: $\omega^{\overline{X}\otimes\overline{Y}}=I_{n_X}\otimes\omega^Y+\omega^X\otimes I_{n_Y}$.

As an application, consider two LI processes, r_t, and g_t, with:We now merge the two previous examples, to incorporate both a time-varying equity premium and a time-varying dividend growth rate. The stochastic discount factor and dividend are given as follows:

---

[7] This subsection probably contains typos.

Example 12 *Stock with decoupled LI processes for the growth rate and the risk premium.*

Consider processes with $dM_t/M_t = -rt - \lambda_s dB_s$, $dD_t/D_t = g_t dt + \sigma dB_t$, where g_t follows the LI process

$$dg_t = -\phi_g \left(g_t - g_*\right) dt - \left(g_t - g_*\right)^2 dt + dN_t^g.$$

The risk premium, $\pi_t = \lambda_t \sigma_t$, follows the LI process:

$$d\pi_t = -\phi_\pi \left(\pi_t - \pi_*\right) dt + \left(\pi_t - \pi_*\right)^2 dt + dN_t^\pi$$

where N_t^g, N_t^π are martingales. Assume that the processes dN_t^g, dN_t^π and dB_t are uncorrelated. Then, the price of a stock, $P_t = E_0\left[\int_0^\infty M_t D_t dt\right]/M_0$, is, by the reasoning of the previous section, $P_t/D_t = E_t\left[\int_{s=t}^\infty \exp\left(-\int_{u=t}^s \left(r + \pi_u - g_u\right) du\right) ds\right]$. In virtue of the above reasoning,

$$E_t\left[\exp\left(\int_t^s -\pi_u + g_u du\right)\right] = E_t\left[\exp\left(\int_t^s -\pi_u du\right)\right] E_t\left[\exp\left(\int_t^s g_u du\right)\right] \tag{47}$$

For general processes, the above equation would in general require the two processes to be independent – for instance, with stochastic volatility, the respective variance processes should be independent. For LI processes, the property required is the weaker $\langle d\pi_t, dg_t\rangle = 0$ for all t's.

Using the values of the LI processes, and integrating, we get, with $R = r + \pi_* - g_*$,[8]

$$P_t/D_t = \frac{1}{R}\left[1 - \frac{\pi_t - \pi_*}{R + \phi_\pi} + \frac{g_t - g_*}{R + \phi_g} - \frac{\left(2R + \phi_\pi + \phi_g\right)\left(\pi_t - \pi_*\right)\left(g_t - g_*\right)}{\left(R + \phi_\pi\right)\left(R + \phi_g\right)\left(R + \phi_\pi + \phi_g\right)}\right]. \tag{48}$$

The central value is again the Gordon formula, $P_t/D_t = 1/R$. It is modified by the current level of the equity premium, and the growth rate of the stock. A stock with a currently high growth rate g_t exhibits a higher price-dividend ratio, and this is amplified when the equity premium is low, as shown by the term $\left(\pi_t - \pi_*\right)\left(g_t - g_*\right)$.

The difference between formula (48) and formula (7) is the here, the processes for π_t and g_t are decoupled, whereas in (7), they where coupled, i.e. in their drift term there was a term $\left(g_t - g_*\right)$. The decoupling forces the presence of a cross term $\left(\pi_t - \pi_*\right)\left(g_t - g_*\right)$ in the expression of the price. In general, one obtains simpler expressions by having one multifactor LI processes, rather than the product of many differnt ones.

The sum of two LI processes is LI. This property is quite trivial, and mentioned for completeness. Suppose two LI process $\left(M_t^i, Y_t^i, \nu^i\right)$, with $M_t^i = \nu^i Y_t^i$, for $i = 1, 2$. Call $d_i = 1 + n_i$ the dimension of Y_t^i, which is 1 plus the number of factors. Then, the SDF $M_t = M_t^1 + N_t^2$ comes from a LI process of dimension $d_1 + d_2$. Indeed, define $Y_t = \left(Y_t^1, Y_t^2\right)$, a vector of dimension $d_1 + d_2$ and $\nu = \left(\nu^1, \nu^2\right)$, and $\Omega = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}$. Then, $E_t\left[Y_{t+1}\right] = \Omega Y_t$, and $M_t = \nu Y_t$.

---

[8] Menzly, Santos and Veronesi (2004, Eq. 20) obtain a similar expression. This is natural because their model belong to the LI class, as Example 7 shows.

6.3 Option pricing with LI processes

One can express transforms of options in the LI framework, under some conditions. As in Duffie, Pan and Singleton (2000), this requires Fourier transforms and ordinary differential equations, but not solving partial differential equations.

Consider the case $D_t = 1$, and the price at time 0 of an option giving at time T the right to buy a bond for a price K. Its price is: $P_t = E_t\left[M_T\left(Z_T\left(X_T, S\right) - K\right)^+\right]$. Given $Z_T\left(X_T, S\right)$ is an affine function of the X_t, write $Z_T\left(X_T, S\right) - K = w' \cdot X_T - K'$, so that the option price at time 0 is:

$$P_0 = E_0\left[M_T\left(Z_T\left(X_T, S\right) - K\right)^+\right] = E_0\left[M_T\left(w' \cdot X_T - K'\right)^+\right] = E_0\left[\left(\psi \cdot Y_T\right)^+\right]$$

with $Y_T = \left(M_t, M_t X_t\right)^\top \in \mathbf{R}^{n+1}$, and $\psi = \left(-K', w\right)^\top \in \mathbf{R}^{n+1}$.

So the problem is solved if we know how to calculate $E_0\left[\left(\psi \cdot Y_T\right)^+\right]$. We can simply transpose the results of Duffie, Pan and Singleton (2000). Assume the following affine process for Y_t, $dY_t = -\omega Y_t dt + dN_t$, where dN_t is a Brownian process with $\left\langle dN_t, dN_t'\right\rangle / dt = 2HY_t$, for with[9] $H \in \mathbf{R}^{(n+1)^3}$. Then, for $\lambda \in \mathbf{C}^{n+1}$, when $E_0\left[e^{\lambda Y_Y}\right]$ is well-defined, one has the following "affine-yield" representation:

$$E_0\left[e^{\lambda' Y_T}\right] = e^{B(T) Y_0} \tag{49}$$

where $B\left(T\right)$ ensures that, with $V\left(T, Y\right) = e^{B(T)Y}$, $\mathcal{A}V - \partial_T V = 0$, which gives:

$$\frac{dB\left(T\right)}{dT} = -B\left(T\right)\omega + B\left(T\right) H B\left(T\right) \tag{50}$$

and $B\left(0\right) = \lambda'$. Typically, the ODE (50) needs to be solved numerically.

We are now done. The knowledge of (49) gives the distribution of Y_T by inversion of the Fourier transform, hence the price of the option.

Decomposition more complicated functions $g\left(X\right)$ on a basis of functions $\left(w' \cdot X_T - K'\right)^+$, one can (in principle) express any option $E_0\left[M_T g\left(X_T\right)\right]$ this way. PDEs are avoided, and replaced by comparatively simpler ODEs and Fourier transforms.

7 Conditions to keep the process well-defined

The results of this paper require that the process be defined for $t \in [0, \infty)$. Appendix A reviews standard sufficient conditions in the one-factor case. The present section present the analogue conditions in the multifactor case. [This section is still being written].

7.1 Practical bottom line

With one factor, the process is well-defined if it stays within $r \leq \overline{r}$, with $\overline{r} < r_* + \phi$. Also, the volatility of the process has to go to 0 near $\overline{r}$. The following is the $n-$factor equivalent. We start with a LI process (31)-(32).

---

[9] H is a tensor, so that HY_t has dimension $(n+1) \times (n+1)$. More explicitly, $\left(HY\right)_{ij} = \sum_k H_{ijk} Y_k$.

Admissibility of the initial conditions

We start from a process $E_t dY_t = -\omega Y_t dt$.

Step 1 – Diagonalization of the process.

Diagonalize the matrix ω, i.e., find q and Δ such that $\omega = q\Delta q^{-1}$, with $\Delta = Diag\left(\delta_1, ..., \delta_{n+1}\right)$, and $\delta_1 \leq ... \leq \delta_{n+1}$. The eigenvector corresponding to eigenvalue δ_j is $\left(q_{ij}\right)_{i=1...n}$.

Define $Q = Diag\left(q_{1j}\right) \cdot q^{-1}$. Then, $\omega = Q^{-1}\Delta Q$, with $(1, .., 1)\, Q = (1, 0, .., 0)$. [10]

Define ∇ ("nabla"), a $(n+1) \times (n+1)$ matrix:[11]

$$\begin{aligned} \nabla_{ij} \quad &: \quad = \left(\delta_{i+1} - \delta_j\right) 1_{i \geq j} \text{ for } i = 1...n &(51)\\ &= \quad 1 \text{ for } i = n+1 &(52) \end{aligned}$$

Step 2 – Admissibility of the initial condition. The initial condition Y_0 should satisfy:

$$\nabla Q Y_0 > 0 \tag{53}$$

where the inequality is meant to hold coordinate by coordinate. Condition (53) is the $n-$dimensional analogue of $r_t - r_* < \phi$ in the one-factor process.

If the initial value of Y_t satisfies (53), and increments are continuous, then all future $Y_{s>t}$ also satisfy (53).

Making the volatility go to zero near the boundaries We consider the region ρ^+

$$\varrho^+ = \left\{Y \in \mathbf{R}^{n+1} \mid \nabla Q Y > 0\right\}$$

As $Y_t^1 = M_t D_t = (0_n, 1) \cdot \nabla Q$, $Y \in \varrho^+$ implies $M_t D_t > 0$.

We define a "killing" function $\kappa : \mathbf{R} \rightarrow \mathbf{R}_+$, such that (i) $\kappa(x) = 0$ for $x \leq 0$; (ii) for x in a right neighborhood of 0, $\kappa(x) = O\left(x^\alpha\right)$, for some $\alpha > 1/2$ and (iii); there is an x_0 (in practice small) such that $\kappa(x) = 1$ for $x > x_0$. Define:

$$K(Y) = \kappa\left(\min_{i=1...n} \frac{\left(\nabla Q Y\right)_i}{\left(\nabla Q Y\right)_{n+1}}\right)$$

That is, $K(Y) = 1$ most of the time, but when Y is close to the boundary of ρ^+, then $K(Y)$ goes to 0.

Transformation of the process to make sure it is defined for $t \in [0, \infty)$.

Start from the "target" process that could be written $d\widetilde{Y}_t = -\omega \widetilde{Y}_t dt + \widetilde{Y}_t dn_t + M_t dN_t$, n_t is a 1-dimensional martingale, N_t a $(n+1)$ dimensional martingale. σ_t captures the log-normal drift in dividend, while dN_t captures innovations to the factors, and $var\left(dN_t\right)/dt$ is bounded. The target process $\widetilde{Y}_t$ might explode in finite time, as in the one-factor process. To stabilize it, define the modified process:

$$dY_t = -\omega Y_t dt + Y_t dn_t + K\left(Y_t\right) M_t dN_t \tag{54}$$

---

[10] If $q_{1j} = 0$ for some j, one just eliminates the space corresponding to eigenvector j, without changing the economics of the process, in particular $M_t D_t$. (To be fleshed out).

[11] The alternative matrix defined by $\nabla_{ij} = 1_{i \geq j}$ also works. It leads to more stringent conditions.

Then, the modified process is defined for $t \in [0, \infty)$. The modified process is well defined, and has correlations identical to those of the initial proces when $K(Y) = 1$, i.e. far enough from the boundary of region ρ^+. The $K(Y)\, dN_t$ term makes the volatility go to 0 when Y is close to the boundary of (53).[12] Otherwise, it is equal to 1. We note that, in practice, the $K(Y_t)$ term will affect the process very rarely.

7.2 Examples

Take 1-dimensional process, $dM_t/M_t = -r_t dt$, $dr_t = \left(-\phi r_t + r_t^2\right) dt + \sigma(r_t)\, dz_t$. Take $Y_t = (M_t, M_t r_t)$. Then, $\omega = \begin{pmatrix} 0 & 1 \\ 0 & \phi \end{pmatrix}$, $Q = \begin{pmatrix} 1 & -1/\phi \\ 0 & 1/\phi \end{pmatrix}$, $\nabla = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and $\nabla Q Y_t = \begin{pmatrix} M_t (1 - r_t/\phi) \\ M_t \end{pmatrix}$. Condition (53), $\nabla Q Y_t > 0$, is equivalent to $r_t < \phi$ and $M_t > 0$. Also, $K(Y_t) = \kappa (1 - r_t/\phi)$. The conditions above implies that it goes to 0 as r_t is in a left neigdhborhood of ϕ.

7.3 Justification

The key lemma is the following.

Lemma 1 *Given a matrix $\omega \in \mathbb{R}^{m \times m}$, with a diagonalization $\omega = Q^{-1} \Delta Q$, with $Q_{1j} = 1$ for $j = 1...m$, and $\Delta = Diag(\delta_0, ..., \delta_{m-1})$. Define:*

$$\begin{aligned} \nabla_{ij} \quad : \quad &= \frac{\delta_i - \delta_{j-1}}{\delta_i - \delta_0} 1_{i \geq j} \text{ for } i = 1...m-1 && (55) \\ &= 1 \text{ for } i = m && (56) \end{aligned}$$

Define $V := \nabla Q \omega Q^{-1} \nabla^{-1}$. Then, for $i < j$, $V_{ij} = 0$, and for $i > j$, $V_{ji} \leq 0$. Also, $V_{ii} = \delta_i$, and $V(1, ..., 1)' = \delta_0 (1, ..., 1)'$. Finally, $(0, .., 0, 1) \nabla Q = (1, 0, ..., 0)$.

Consider then $Z_t = \nabla Q Y_t$. We have $E_t dZ_t/dt = -V Z_t$, which has non-negative non-diagonal elements. Hence, an element $Z_t^i = 0$, while $Z_t^j > 0$, then $E dZ_t^i/dt \geq 0$. This means that, in the deterministic version of the process, if $Z_0 > 0$, then for all $t > 0, Z_t > 0$.

In discrete time, we suppose that Ω has positive eigenvalues. We start from $Y_{t+1} = \Omega Y_t$, and call $\omega = I_{n+1} - \Omega$. $Z_t = \nabla Q Y_t$. We have $Z_{t+1} = K Z_t$, with $K = I_{n+1} - \nabla Q \omega Q^{-1} \nabla^{-1}$. K which has weakly positive non-diagonal elements, and as diagonal elements, the eigenvalues of Ω, so that finally K has weakly positive coefficients. Hence, if $Z_t \geq 0$, $Z_{t+1} \geq 0$.

8 Conclusion

Linearity-inducing processes are very tractable, as they yield closed forms for stocks and bonds, and prices that are linear in factors. They are likely to be useful in several parts of economics, when trend growth rates, or risk premia, are time-varying.

The results of this paper suggest the following questions.

---

[12] The above procedure works with continuous increments. When there are jumps, the jumps should not transport Y_t outside of ρ^+.

First, it would be desirable to study explicit, non-toy, economic models that take advantage of the tractability offered by the LI structure.

Second, the LI processes being defined by moment conditions (Eq. 13-14), they lend themselves to estimation and testing by GMM techniques.

Third, LI processes suggest a new way to linearize models. Given a model, one could do a Taylor expansion expressing moments $E_t\left[m_{t+1}\right]$ and $E_t\left[m_{t+1}Y_{t+1}\right]$ as a linear function of the factors, thereby making equations 13-14 hold to a first order approximation. The projected model is then in the LI class, and its asset prices are approximations of the prices of the initial problem. Hence the LI class offers a way to derive linear approximations of the asset prices of more complicated models.

Fourth, the LI class suggests a way to create further discount factor processes. The background state vector Y_t could follow a process richer than an autoregressive process, and the stochastic discount factor, which simply a linear projection of the state vector in LI processes, could be a richer function of it.

We conclude that LI processes might be a useful addition to the economists' toolbox.

Appendix A. Regularity conditions for the one-factor process

This appendix details conditions for the existence and uniqueness of the solutions. We recommend Katatzas and Shreve (1991 Chapter 5.5) and Revuz and Yor (1999, Chapter IX) for systematic treatments, and Ait-Sahalia (1996, Appendix; 2002) and Duffie (2001, Appendix E) for pedagogical overviews. We call $\mathcal{D} = (\underline{r}, \overline{r})$ the domain of existence of r, and c an arbitrary point in $\mathcal{D}$. We call $\mu(r)$ the drift of r, and assume $dN_t = \sigma(r_t)\, dz_t$. We make the following assumptions.

(i) The drift and diffusion functions are continuously differentiable in r in $\mathcal{D}$, and $\sigma^2(r) > 0$ in $\mathcal{D}$.

(ii) The integral of $m(r) = \exp\left(\int_c^r 2\mu(u)/\sigma^2(u)\, du\right)/\sigma^2(r)$ converges at both boundaries of $\mathcal{D}$.

(iii) The integral of $s(r) = \exp\left(-\int_c^r 2\mu(u)/\sigma^2(u)\, du\right)$ diverges at both boundaries of $\mathcal{D}$.

(iv) μ is Lipschitz continuous, and there is a function $\rho(x) : \mathbf{R}_+ \to \mathbf{R}_+$, with $\rho(0) = 0$, such that for any $\varepsilon > 0$, $\int_{(0,\varepsilon)} \rho(x)^{-2}\, dx = +\infty$, and $|\sigma(x) - \sigma(y)| \leq \rho(|x-y|)$.

If conditions (i)-(iv) are satisfied, then there is a unique Ito process $\{r_t, t \geq 0\}$ which is a strong solution of the stochastic differential equation (28) with initial condition $r_0 = r$. Moreover, $\{r_t, t \geq 0\}$ is Markov.

The key substantive point is that the process is defined for all $t \geq 0$, and does not explode. This condition is crucial, as if we started with $r_0 > \beta$, the process would explode in finite time with positive probability, so that the process would not be defined for all times.

Conditions (i), (ii) and (iv) guarantee the existence and uniqueness of the solution up to the variable may hit the boundaries. Condition (iii) implies that the boundaries are actually not reached. The intuition is as follows. Consider the correct boundary. Condition (iii) implies $\mu(\overline{r}) < 0$, so that the process tends to return inside $\mathcal{D}$, and also requires that $\sigma^2(r)$ tends to 0 fast enough as $r \uparrow \overline{r}$.

Sufficient conditions to ensure (i)-(iv) Conditions (i) and (ii) guarantee that the stochastic differential equation (28) admits a unique strong solution. Those conditions are verified in the following cases. Condition (iii) guarantees that the end points $\mathcal{D}$ of are natural boundaries.

We assume $\mu(\overline{r}) < 0$ and $\lim_{r \to \underline{r}} \mu(r) > 0$, so that close to the end points of $\mathcal{D}$, the process tends to go back inside $\mathcal{D}$. In the case $\mu(r) = (r-\alpha)(r-\beta)$, with $\alpha < \beta$, this corresponds to $\overline{r} \in (\alpha, \beta)$ and $\overline{r} \in [-\infty, \alpha)$.

Conditions (ii) and (iii) are verified if the following conditions (C-$\mathcal{D}$) hold. For r in a left-neighborhood of $\overline{r}$, $\sigma^2(r) \sim k(\overline{r} - r)^\kappa$, with ($\kappa > 1$ and $k > 0$) or ($\kappa = 1$ and $0 < k < -2m(\overline{r})$). If $\underline{r} > -\infty$, for r in a right neighborhood of $\underline{r}$, $\sigma^2(r) \sim k'(r - \underline{r})^{\kappa'}$, with ($\kappa' > 1$ and $k' > 0$) or ($\kappa' = 1$ and $0 < k' < 2m(\underline{r})$). If $\underline{r} = -\infty$, then $\underline{r}$ is a natural boundary if, for r in a neighborhood of $-\infty$ $\sigma^2(r) \sim k|r|^\beta$, with $k > 0$ and $\beta < 3$. Those last conditions imply assumptions (ii), (iii).

For $\underline{r} = -\infty$, the situation is complex for condition (iv), as the standard conditions found in textbooks do not apply. $\mu(r)$ is not Lipschitz continuous, as $\mu'(r)$ is unbounded. We conjecture that a simple weakening of condition (iv) will allow the case $\underline{r} = -\infty$.

If $\underline{r} > -\infty$, the above conditions (C-$\mathcal{D}$) also imply (iv), as one can take $\rho(x) = K \max\left(x^{\kappa/2}, x^{\kappa'/2}, x\right)$, for a large enough constant K.

Appendix B. Proofs

In some of the proofs, we will use the following Lemmas, which are standard facts. For a review, see, e.g., Shreve (2005).

Lemma 2 *With* $a \in \mathbb{R}, b, c \in \mathbb{R}^n$, *and* $d \in \mathbb{R}^{n^2}$, *suppose that* $a - b'd^{-1}c \neq 0$ *and* d *is invertible. Then the* $(n+1) \times (n+1)$*-dimensional matrix* $\begin{pmatrix} a & b' \\ c & d \end{pmatrix}$ *is invertible, and its inverse is:*

$$\begin{pmatrix} a & b' \\ c & d \end{pmatrix}^{-1} = \frac{1}{a - b'd^{-1}c} \begin{pmatrix} 1 & -b'd^{-1} \\ -d^{-1}c & ad^{-1} \end{pmatrix} \tag{57}$$

In the above equation, $a - b'd^{-1}c$ is a real number.

Lemma 3 *With* $n \in \mathbb{N}_+$, $t \in \mathbb{N}$, $a \in \mathbb{R}, b \in \mathbb{R}^n$, *and* $d \in \mathbb{R}^{n^2}$. *Call* $0_{n\times1}$ *is the zero* $n \times 1$ *matrix made of 0's, and suppose that* $(aI_n - d)$ *is invertible. Then*

$$\exp\left[\begin{pmatrix} a & b' \\ 0_{n\times1} & d \end{pmatrix}\right] = \begin{pmatrix} e^a & b'\left(e^a I_n - e^d\right)(aI_n - d)^{-1} \\ 0_{n\times1} & e^{dt} \end{pmatrix} \tag{58}$$

$$\begin{pmatrix} a & b' \\ 0_{n\times1} & d \end{pmatrix}^t = \begin{pmatrix} a^t & b'\left(a^t I_n - d^t\right)(aI_n - d)^{-1} \\ 0_{n\times1} & d^t \end{pmatrix} \tag{59}$$

Proof of Theorem 1 Recall (16), $E_t[Y_{t+1}] = \Omega Y_t$. Iterating, it implies that for all

$$\forall T \geq 0, \forall t, E_t[Y_{t+T}] = \Omega^T Y_t \tag{60}$$

Hence, using the definition of the zero-coupon (10),

$$M_t D_t Z_t(T) = E_t[M_{t+T} D_{t+T}] = E_t\left[\begin{pmatrix} 1 & 0 \end{pmatrix} Y_{t+T}\right] = \begin{pmatrix} 1 & 0 \end{pmatrix} E_t[Y_{t+T}] = \begin{pmatrix} 1 & 0 \end{pmatrix} \Omega^T Y_t$$

i.e., dividing by $M_t D_t$,

$$Z_t = \begin{pmatrix} 1 & 0 \end{pmatrix} \Omega^T (M_t D_t)^{-1} Y_t = \begin{pmatrix} 1 & 0 \end{pmatrix} \Omega^T \begin{pmatrix} 1 \\ X_t \end{pmatrix}$$

i.d. Eq. 60.

The formula for $\gamma = 0$ comes from 3.

Proof of Theorem 2 Because the Theorem arises often in practice, it is useful to have different ways to derive it.

The abstract proof . We use (18), which gives the perpetuity price:

$$V_t = \sum_{T=0}^{\infty} Z_t(T) = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \left(\sum_{T=0}^{\infty} \Omega^T\right) \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot (I_n - \Omega)^{-1} \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix}$$

$\sum_{T=0}^{\infty} \Omega^T$ is summable because all eigenvalues of Ω have a modulus less than 1. We use Lemma 2 to calculate $(I_n - \Omega)^{-1}$, and conclude.

A plug-and-verify derivation. We seek a solution of the type $V_t = c - 1 + h' X_t$, which we know exists, by integration of (36). The arbitrage equation is:

$$V_t = 1 + E\left[m_{t+1} V_{t+1}\right]$$

i.e.

$$\begin{aligned} c + h' X_t &= 1 + E\left[m_{t+1}\left(c + h' X_t\right)\right] = 1 + c\left(\alpha + \delta' X_t\right) + h'\left(\gamma + \Gamma Y_t\right) \\ &= \left[1 + c\alpha + h'\gamma\right] + \left[c\delta' + h'\Gamma\right] X_t \end{aligned}$$

i.e.

$$\begin{aligned} c &= 1 + c\alpha + h'\gamma \\ h' &= c\delta' + h'\Gamma \end{aligned}$$

The last equation gives $h' = c\delta' \left(1 - \Gamma\right)^{-1}$, and plugging in the first equation we get $c\left[1 - \alpha - \delta' \left(1 - \Gamma\right)^{-1} \gamma\right] = 1$, which yields the desired result.

Proof of Proposition 1 Here we present a simple, calculational proof. It turns out that the proof of the $n-$dimensional case, in section 5, leads to a simpler proof, though a more abstract one.

By shifting r_t by a constant, is sufficient to prove the Theorem in the case $r_* = 0$. Indeed, in the general case, define $r'_t = r_t - r_*$, which satisfies (28) with $r_* = 0$. As $E_0\left[\exp\left(-\int_0^T r_s ds\right)\right] = e^{-r_* T} E_0\left[\exp\left(-\int_0^T r'_s ds\right)\right]$, the general case deduces from the case $r_* = 0$.

It is sufficient to prove the statement for an initial time equal to 0. We fix $T > 0$ and define $V_t = E_t\left[\exp\left(-\int_t^T r_s ds\right)\right]$, for $t \in [0, T]$. The arbitrage equation of V_t is:

$$-r_t V_t + E\left[dV_t\right]/dt = 0 \tag{61}$$

We call A the infinitesimal diffusion operator of the process, which in the case $dN_t = \sigma\left(r_t\right) dz_t$ is:

$$\mathcal{A} f\left(r\right) = \left(-\phi r + r^2\right) f'\left(r\right) + \frac{\sigma^2\left(r\right)}{2} f''\left(r\right)$$

Given r_t is the state variable, the solution is of the type $V_t = W_t = w\left(r_t, t\right)$. The PDE (61) becomes:

$$-rw\left(r, t\right) + \mathcal{A} w\left(r, t\right) + \frac{\partial}{\partial t} w\left(w, t\right) = 0$$

We seek a solution with the functional form:

$$w\left(r, t\right) = A\left(t\right) r + B\left(t\right) \tag{62}$$

The boundary condition $W_T = 1$ implies $A(T) = 0$ and $B(T) = 1$. The PDE is $Q = 0$ with:

$$\begin{aligned} Q &= -rw(r,t) + \left(-\phi r + r^2\right) \frac{\partial}{\partial r} w(r,t) + \frac{\sigma^2(r)}{2} \frac{\partial^2}{\partial r^2} w(r,t) + \frac{\partial}{\partial t} w(w,t) \\ &= -r\left[A(t)r + B(t)\right] + \left(-\phi r + r^2\right) A(t) + \frac{\sigma^2(r)}{2} \cdot 0 + \left[A'(t)r + B'(t)\right] \\ &= \left[B'(t)\right] + r\left[-B(t) - \phi A(t) + A'(t)\right] + r^2\left[-A(t) + A(t)\right] \end{aligned} \tag{63}$$

Importantly, the r^2 terms, $-A(t) + A(t)$, cancel out. This explains why adding the quadratic term in the drift allows of a functional form (62) to work. Also, because $\partial^2 w/\partial r^2 = 0$ in the functional form (62), the Ito second-order term (equal to $\frac{\sigma^2(r)}{2}\partial^2 w/\partial r^2$ in the case of a diffusion) is 0, which explains how the $\sigma^2(r)$ can not appear in the final solution.

The PDF $Q = 0$ in (63) is satisfied for all r if and only if $B'(t) = 0$ (which in virtue of $B(T) = 1$ implies $B(t) = 1$ for all t's), and

$$-B(t) - \phi A(t) + A'(t) = 0 \tag{64}$$

whose solution is $A(t) = -1/\phi + ce^{\phi t}$, for a constant c, determined by $A(T) = 0$, which gives $A(t) = \left(e^{-\phi(T-t)} - 1\right)/\phi$. In the end, we obtain:

$$W_t = A(t) r_t + B(t) = 1 + \frac{e^{-\phi(T-t)} - 1}{\phi} r_t$$

The above shows that W_t satisfies $-r_t W_t + E[dW_t]/dt = 0$, and $W_T = 1$, but not yet that it is the equal to the bond price. To conclude the proof, we define $L_t = \exp\left(-\int_0^t r_s ds\right) W_t$. As $E_t[dW_t - r_t W_t dt] = 0$, we have $E_t[dL_t] = E_t\left[\exp\left(-\int_0^t r_s ds\right)(dW_t - r_t W_t dt)\right] = 0$. Hence L_t is a martingale, which implies, $L_0 = E_0[L_T]$, i.e. $W_0 = E_0\left[\exp\left(-\int_0^T r_s ds\right) W_T\right] = E_0\left[\exp\left(-\int_0^T r_s ds\right)\right]$, i.e.

$$1 + \frac{e^{-\phi(T-t)} - 1}{\phi} r_t = E_0\left[\exp\left(-\int_0^T r_s ds\right)\right]$$

which is the statement of the Proposition.

Proof of Theorem 3 Recall the definion of ω in (34).

Compact proof. $E_t[d(Y_t)]/dt = -\omega Y_t$. It is well-known that this implies:

$$\forall T \geq 0, E_t[Y_{t+T}] = e^{-\omega T} Y_t. \tag{65}$$

Indeed, to prove (65) in the case $t = 0$ (which is enough), set $T > 0$, and define $z_t = e^{\omega(t-T)} Y_t$. Then,

$$E_t[dz_t] = E_t d\left(e^{\omega(t-T)} Y_t\right) = E_t\left[d\left(e^{\omega(t-T)}\right)\right] Y_t + e^{\omega(t-T)} E_t[d(Y_t)] = \left[e^{\omega(t-T)} \omega Y_t dt + e^{\omega(t-T)}(-\omega Y_t) dt\right] = 0$$

Hence z_t is a martingale, and $E_0[z_T] = z_0$, i.e.$E_0[Y_T] = e^{-\omega T} Y_0$.

Given.(65) and $M_s = \left(\begin{array}{cc} 1 & 0 \end{array} \right) Y_s$,

$$Z_t(T) = E_t[M_{t+T}] = E_t\left[\left(\begin{array}{cc} 1 & 0 \end{array} \right) Y_{t+T}\right] = \left(\begin{array}{cc} 1 & 0 \end{array} \right) E_t[Y_{t+T}] = \left(\begin{array}{cc} 1 & 0 \end{array} \right) e^{-\omega T} Y_t.$$

The formula for $b = 0$ comes from Lemma 3.

A "plug and verify" proof that is easier to replicate in applied contexts. We guess the functional form $Z(X_t, T) = A(T) + \gamma'(T) X_t$, for a functions to be determined $A(T)$ and $B(T)$ with values in $\mathbb{R}$ and $\mathbb{R}^n$ respectively. With this functional form, $E[dZ_t/dt] = \gamma'(T)(b - \Phi X_t + r_t X_t) - \partial_T Z(X_t, T)$, so that the PDE becomes:

$$0 = -\left(r_* + \beta' X_t\right)\left(A(T) + \gamma'(T) X_t\right) + \gamma'(T)\left(b - \Phi X_t + \left(\beta' X_t\right) X_t\right) - \frac{d}{dT} A(T) - \frac{d}{dT}\gamma'(T) X_t$$

which holds for all X_t's if and only iff:

$$\begin{aligned} \frac{d}{dT} A(T) &= -r_* A(T) + \gamma'(T) b \\ \frac{d}{dT}\gamma'(T) &= -\beta' A(T) - \gamma'(T)(\Phi + r_*) \end{aligned}$$

which in matrix form writes:

$$\frac{d}{dT}\left(\begin{array}{cc} A(T) & \gamma'(T) \end{array} \right) = -\left(\begin{array}{cc} A(T) & \gamma'(T) \end{array} \right)\omega$$

which solves

$$\left(\begin{array}{cc} A(T) & \gamma'(T) \end{array} \right) = \left(\begin{array}{cc} A(0) & \gamma'(0) \end{array} \right)\exp[-\omega t] = \left(\begin{array}{cc} 1 & 0 \end{array} \right)\exp[-\omega t]$$

The bond price is $Z_t(T) = \left(\begin{array}{cc} A(T) & \gamma'(T) \end{array} \right)\left(\begin{array}{c} 1 \\ X_t \end{array} \right)$, which completes the proof.

Simple proof when $b = 0$. It is enough to consider $r_* = 0$, by redefining r_t to be $r_t - r_*$. We consider the $Z(X_t, T)$ the price of zero-coupon bond of maturity T. The arbitrage equation for Z is:

$$0 = -r_t Z + E[dZ_t]/dt.$$

We guess the functional form $Z(X_t, T) = 1 + \gamma'(T) X_t$, for a function $\gamma(T)$ with values in $\mathbb{R}^n$ to be determined. With this functional form, $E[dZ_t/dt] = \gamma'(T)(-\Phi X_t + r_t X_t) - \partial_T Z(X_t, T)$, so that the PDE becomes:

$$\begin{aligned} 0 &= -r_t\left(1 + \gamma'(T) X_t\right) + \gamma'(T)(-\Phi X_t + r_t X_t) - \frac{d}{dT}\gamma(T)' X_t \\ &= -r_t - \gamma'(T)\Phi X_t - \frac{d}{dt}\gamma(T)' X_t. \end{aligned}$$

The $r_t X_t$ terms cancel out, which motivated their inclusion in the LI process. Using $r_t = \beta' X_t$, we obtain:

$$0 = -\beta' - \gamma'(T)\Phi - \frac{d}{dt}\gamma(T)' = 0 \Leftrightarrow \frac{d}{dt}\gamma(T)' = -\gamma'(T)\Phi - \beta'.$$

One integrates this matrix equation like a scalar equation, using the boundary condition $\gamma(0) = 0$, which gives:

$$\gamma(T)' = \beta' \frac{e^{-\Phi T} - 1}{\Phi}$$

and finally the earlier expression for the bond price.

Proof of Theorem 4 *A compact proof.* We use (36). The perpetuity price is:

$$V_t = \int_0^\infty Z_t(T) = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \left(\int_0^\infty e^{-\omega T} dT \right) \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \omega^{-1} \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix}$$

We use the Lemma 2 to calculate ω^{-1}, and conclude.

A plug-and-verify derivation. We seek a solution of the type $V_t = c + h' X_t$, which we know exists, by integration of (36). The PDE is: $1 - r_t V_t + E[dV_t]/dt = 0$, i.e.

$$1 - \left(r_* + \beta' X_t\right)\left(c + h' X_t\right) + h' \left[b - \Phi X_t + \left(\beta' X_t\right) X_t\right] = 0$$

This is satisfies if and only if the constant and the term in X_t are zero:

$$\begin{aligned} r_* h' + \beta' c + h' \Phi &= 0 \\ 1 - r_* c + h' b &= 0 \end{aligned}$$

hence $h' = -\beta' c (r_* + \Phi)^{-1}$ and $1 - c\left[r_* + \beta' (r_* + \Phi)^{-1} b\right]$, which gives $c = 1/\left[r_* + \beta' (r_* + \Phi)^{-1} b\right]$, and proves the Proposition.

A simple proof, valid when $b = 0$. We integrate (37):

$$\begin{aligned} P_t &= \int_0^\infty Z_t(T)\, dT = \int_0^\infty e^{-r_* T} dT + \int_0^\infty e^{-r_* T} \beta' \frac{e^{-\Phi T} - 1}{\Phi} X_t dT \\ &= \frac{1}{r_*} + \beta' \left(\int_0^\infty e^{-(r_* + \Phi)T} - e^{-r_* T} dT \right) \frac{1}{\Phi} X_t \\ &= \frac{1}{r_*} + \beta' \left(\frac{1}{r_* + \Phi} - \frac{1}{r_*} \right) \frac{1}{\Phi} X_t = \frac{1}{r_*} - \beta' \frac{1}{r_* (r_* + \Phi)} X_t. \end{aligned}$$

Proof of Proposition 3 It is as the proof of Theorem 3. Given.(65) and $M_s D_s X_s = \begin{pmatrix} 0 & 1 \end{pmatrix} Y_s$,

$$\begin{aligned} E_t\left[M_{t+T} D_{t+T} X_{t+T}\right] &= E_t\left[\begin{pmatrix} 0 & 1 \end{pmatrix} Y_{t+T}\right] = \begin{pmatrix} 0 & 1 \end{pmatrix} E_t\left[Y_{t+T}\right] \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} e^{-\omega T} Y_t = \begin{pmatrix} 0 & 1 \end{pmatrix} e^{-\omega T} \begin{pmatrix} M_t D_t \\ M_t D_t X_t \end{pmatrix} \end{aligned}$$

i.e. (39). Formula (40) comes from Lemma 3.

Proof of Proposition 4 It is as the proof of Theorem 4. Call V_t the left-hand side of (41). As for all s, $M_s D_s X_s = \begin{pmatrix} 0 & 1 \end{pmatrix} Y_s$

$$\begin{aligned}
M_t D_t V_t &= E_t\left[\int_0^\infty M_{t+T} D_{t+} X_{t+T} ds\right] = E_t\left[\int_0^\infty \begin{pmatrix} 0 & 1 \end{pmatrix} Y_{t+T} dT\right] = \begin{pmatrix} 0 & 1 \end{pmatrix} \int_0^\infty E_t\left[Y_{t+T}\right] dT \\
&= \begin{pmatrix} 0 & 1 \end{pmatrix} \int_0^\infty e^{-\omega T} Y_t dT, \text{ by Eq. 65} \\
&= \begin{pmatrix} 0 & 1 \end{pmatrix} \left(\int_0^\infty e^{-\omega T} dT\right) Y_t = \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \omega^{-1} \cdot \begin{pmatrix} M_t D_t \\ M_t D_t X_t \end{pmatrix}
\end{aligned}$$

so that

$$V_t = \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \omega^{-1} \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix}$$

We use the Lemma 2.We calculate ω^{-1} and conclude.

Also, one can derive the Proposition by solving for $V_t = A + BX_t$ satisfying $E_t\left[d\left(M_t D_t V_t\right)\right] / \left(M_t D_t\right) + X_t dt = 0$.

Appendix C. A class of processes admitting a closed form for perpetuities

The following Proposition shows a way to generate processes that have closed form for perpetuities. However, typically that functional process does not yield a closed form for finite-maturity claims, unlike LI processes.

Proposition 5 *(Class of processes admitting a closed form for perpetuities) Suppose that there is a process* ψ_t*, and constants* α *and* β *such that*

$$d\psi_t = \left(r_t \psi_t + \alpha r_t - \beta\right) dt + dN_t,$$

where dN_t *is an adapted martingale, and is essentially arbitrary except for technical conditions. Then:*

$$V_t = \frac{\psi_t + \alpha}{\beta} \tag{66}$$

is a solution of the perpetuity PDE: $1 - r_t V_t + E\left[dV_t\right]/dt = 0$*. If* dN_t *satisfies regularity conditions that make the process well-defined, then* V_t *is the price of a perpetuity,* $V_t = E_t\left[\int_t^\infty e^{-\int_t^s r_u du} ds\right]$.

Proof. Call $V_t = \left(\psi_t + \alpha\right)/\beta$.

$$1 - r_t V_t + E\left[dV_t\right]/dt = 1 - r_t \frac{\psi_t + \alpha}{\beta} + \frac{1}{\beta}\left(r_t \psi_t + \alpha r_t - \beta\right) = 0.$$

■

A first example is the LI process: $\psi(r) = r$, $\alpha = -\left(\lambda + \mu\right), b = -\lambda\mu$. Eq. 66 gives $V_t = \left(\lambda + \mu - r_t\right) / \left(\lambda\mu\right)$, the formula (30) for perpetuities.

A second example is

$$d\left(1/r_t\right) = \phi\left(r_t - r^*\right)dt + dN_t$$

where dN_t ensures $r_t > 0$, i.e., r_t mean-reverts to a central value r^*. This yields a price for perpetuities equal to:

$$P_t = \left(\frac{1}{r_t} + \frac{\phi}{r^*}\right) / \left(1 + \phi\right) \tag{67}$$

by applying Proposition 5 to $\psi\left(r\right) = 1/r$, $\alpha = \phi, b = 1 + \phi r^*$. The formula (67) admits the following interpretation. If r_t remains at its current value, the perpetuity price would be $1/r_t$. If r_t was at its long run central value r^*, the perpetuity price would be $1/r^*$. In general, the perpetuity price (67) is a simple weighted sum of those two perpetuity values. ϕ indicates the speed of mean reversion, and gives the weight between the "current value" $1/r_t$ and the "long run value" $1/r^*$.

Appendix D. Approximating non-LI processes with LI processes

LI processes offer a way to approximate the price of stocks and bonds with non-LI processes, often to an arbitrary degree of precision. This Appendix illustrates this in the example of section 2.1, where the stock dividend growth (detrended) follows an OU process : $dg_t = -\phi g_t dt + \sigma dz_t$. The general properties of approximation with LI processes would require a full paper, but the present appendix simply illustrates that a preliminary investigation justifies being optimistic.

First-order approximation We return to the model of section ection 2.1, with $R = r - g_*$, and here we call $g_t = \gamma_t$. Define $Y_t^1 = e^{-Rt}D_t$, and $Y_t^2 = e^{-Rt}D_t g_t$. We have: $E_t\left[dY_{1,t}\right]/dt = \left(-R + g_t\right)Y_{1,t} = -RY_{1,t} + Y_{2,t}$ and

$$dY_{2,t} = Y_{1,t}\left(-\left(\phi + R\right)g_t + g_t^2\right)$$

To approximate g_t^2, we replace it by its steady state mean. To find it, we observing that $E_t\left[dg_t^2\right]/dt = -2\phi g_t^2 + \sigma^2$, so that taking the expectation at time 0, we get $\lim_{t\to\infty} E_0\left[g_t^2\right] = \sigma^2/\left(2\phi\right)$. Hence we approximate $dY_{2,t} \simeq Y_{1,t}\left(-\left(\phi + R\right)g_t + \sigma^2/\left(2\phi\right)\right)$. Hence we approximate Y_t by Y_t^*, where

$$E_t\left[dY_t^*\right]/dt = -\begin{pmatrix} R & -1 \\ -\sigma^2/\left(2\phi\right) & R + \phi \end{pmatrix} Y_t^*$$

Applying Theorem 4, we get:

$$P_t^*/D_t = \frac{g_t + R + \phi}{R\left(R + \phi\right) - \sigma^2/\left(2\phi\right)} \tag{68}$$

Insert Figure 3 about here

Figure 3 plots the LI approximation, and the exact expression. We find only a small discrepancy (less than 1.5%) between the two expressions. We conclusion is that the first order approximation of the OU process by a LI process will be rather good, and useful for theoretical purposes.

If the goal is high-level numerical accuracy, we turn to an approximation of arbitrary order.

Approximation of arbitrary order In some examples, and perhaps virtually always (at least, when the processes defining the functions are analytic), it is possible to make LI processes approximate the prices of non-LI process to an arbitrary degree of precision. We provide a simple illustration of this. Define

$$Y_{it} = e^{-rt} D_t g_t^{i-1}$$

for $i = 1, 2, ...$ Hence, the vector of factors is $X_t = \left(g_t, g_t^2, g_t^3, ...\right)$. [13] We have:

$$\begin{aligned} E_t \left[dY_{i,t}\right]/dt &= e^{-rt} D_t \left(g_t^{i+1} + (i-1)(-\phi)\, g_t^i + (i-1)(i-2)\frac{\sigma^2}{2} g_t^{i-3} \right) - rY_{i,t} \\ &= (i-1)(i-2)\frac{\sigma^2}{2} Y_{i-2,t} - \left[r + (i-1)\phi\right] Y_{i,t} + Y_{i+1,t} \end{aligned}$$

so that $E_t \left[dY_t\right] = -\omega Y_t dt$, with $\omega_{i,i-2} = -(i-1)(i-2)\sigma^2/2$,$\omega_{i,i} = r + (i-1)\phi, \omega_{i,i+1} = -1$ and $\omega_{ij} = 0$ otherwise. So the price is:

$$P_t/D_t = (1, 0, ..., 0, ..)\, \omega^{-1} \left(1, g_t, g_t^2, ..., g_t^n, ...\right) \tag{69}$$

The sum can be truncated up to step n, i.e be take to be the restriction of the vector to the first n dimensions. We compare the LI (69) to the exact expression (2). Numerical results, reported in Figure 3, show that the approximation is very good, even for $n = 5$.

Insert Figure 4 about here

It would be good to generalize the above procedure, probably in a future paper. It suggests that LI processes allow to evaluate the price of many non-LI processes (e.g., those with analytic expansions), to an arbitrary degree of precision.

References

Abel, Andrew (2007) "Equity Premia with Benchmark Levels of Consumption: Closed-Form Results," in Rajnish Mehra and Edward Prescott (eds.), *Handbook of Investments: Equity Premium.*

Ait-Sahalia, Yacine (1996): "Testing Continuous-Time Models of the Spot Interest Rate," *Review of Financial Studies*, 9, 385-426.

Ait-Sahalia, Yacine (2002): "Maximum Likelihood Estimation of Discretely Sampled Diffusions: A Closed-Form Approximation Approach," *Econometrica*, 70, 223-262.

Ahn, Dong-Hyun, Robert F. Dittmar, and A. Ronald Gallant (2002), "Quadratic Term Structure Models: Theory and Evidence," *Review of Financial Studies,* 15,. 243-288.

Ang, Andrew and Jun Liu, "How to Discount Cashflows with Time-Varying Expected Returns," 2004, *Journal of Finance*, 59, 6, 2745-2783.

---

[13] Of course, the same reasoning could be done with another basis $f_i(g_t)$ for the transforms of g_t.

Bansal, Ravi and Amir Yaron (2004): “Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles,” *Journal of Finance* 59, 1481-1509.

Barro, Robert (2006): “Rare Disasters and Asset Markets in the Twentieth Century,” *Quarterly Journal of Economics*, 121, forthcoming.

Bhattacharya, S. (1978). “Project valuation with mean reverting cash flow streams”, *Journal of Finance*, vol. 33, pp. 1317–31.

Bekaert, Geert, Eric Engstrom and Steve Grenadier (2005) “Stock and Bond Returns with Moody Investors,” Columbia University, Working Paper.

Brigo, D., and F. Mercurio (2001): *Interest Rate Models, Theory and Practice.* Berlin; New York: Springer.

Brennan, Michael J., and Schwartz, E.S., “A Continuous Time Approach to the Pricing of Bonds,” *Journal of Banking and Finance*, 3, 2, 133-155, (July 1979).

Brown, S. J., W. N. Goetzmann, and S. A. Ross (1995): “Survival,” *Journal of Finance*, 50, 853-873.

Campbell, John Y., and John Cochrane (1999): “By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior,” *Journal of Political Economy*, 107, 205-251.

Campbell, John Y., and Robert J. Shiller (1988): “The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors,” *Review of Financial Studies* 1, 195-228.

Cochrane, John (2006): “The Dog That Did Not Bark: A Defense of Return Predictability,” NBER Working Paper 12026.

Cochrane, John and Monika Piazzesi (2005) “Bond risk premia,” *American Economic Review* Volume 95(1), pp. 138-160.

Collin-Dufresne, Pierre, and R. S. Goldstein (2002): “Do Bonds Span the Fixed Income Markets? Theory and Evidence for Unspanned Stochastic Volatility,” *Journal of Finance*, 57, 1685-1730.

Constantinides, George M., “A theory of the nominal term structure of interest rates,” Review of Financial Studies, 1992, 5, 531-552

Cox, John C., J. E. Ingersoll, Jr., and Stephen A. Ross (1985): “A Theory of the Term Structure of Interest Rates,” *Econometrica*, 53, 385-407.

Croce, M., Martin Lettau, and Sydney Ludvigson (2006): “Investor Information, Long-Run Risk, and the Duration of Risky Cash Flows,” Mimeo, New York University.

Driessen, J., P. Maenhout, and G. Vilkov (2005): “Option-Implied Correlations and the Price of Correlation Risk,” Working Paper, INSEAD.

Duffie, D. (2001): *Dynamic Asset Pricing Theory.* Princeton: Princeton University Press.

Duffie, D., and R. Kan (1996): “A Yield-Factor Model of Interest Rates,” *Mathematical Finance*, 6, 379-406.

Duffie, D., J. Pan, and K. Singleton (2000): "Transform Analysis and Asset Pricing for Affine Jump Diffusions," *Econometrica*, 68, 1343-1376.

El Karoui, N., Myneni, R., and Viswanathan, R. (1992) Arbitrage Pricing and Hedging of Interest Rate Claims With State variables: I theory, working paper University of Paris.

Fama, E., and K. R. French (1996): "Multifactor Explanations of Asset Pricing Anomalies," *The Journal of Finance*, 51, 55-84.

Gabaix, X., and D. Laibson (2002): "The 6D Bias and the Equity Premium Puzzle," in *NBER Macroeconomics Annual Year 2001.* ed. by B. Bernanke and K. Rogoff. Cambridge: MIT Press, 257-312.

Gabaix, X., P. Gopikrishnan, V. Plerou, and H. E. Stanley (2003): "A Theory of Power Law Distributions in Financial Market Fluctuations, *Nature*, 423, 267–230.

Gabaix, X., P. Gopikrishnan, V. Plerou, and H. E. Stanley (2006): "Institutional Investors and Stock Market Volatility," *Quarterly Journal of Economics*, 121, 461–504.

Hansen, Lars, John C. Heaton, and N. Li (2005): "Consumption Strikes Back? Measuring Long Run Risk," Mimeo, University of Chicago.

Hansen, Lars, and José Scheinkman (2006): "Long Term Risk: An Operator Approach," Mimeo, Princeton University.

Heath, D., R. Jarrow, and A. Morton (1992): "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, 60, 77-105.

Julliard, Christian, and Jonathan Parker (2005): "Consumption Risk and the Cross-Section of Expected Returns," *Journal of Political Economy*, 113, 185-222.

Karlin, S., and H. M. Taylor (1981): *A Second Course in Stochastic Processes.* New York; London: Academic Press.

Leippold, Markus and Liuren Wu.(2002) "Asset pricing under the quadratic class," *The journal of financial and quantitative analysis*, 37, 271-295.

Lettau, Martin and Jessica Wachter (2007), "Why is Long-Horizon Equity Less Risky? A Duration-Based Explanation of the Value Premium", *Journal of Finance.*

Litterman, R., and José Scheinkman (1991): "Common Factors Affecting Bond Returns," *Journal of Fixed Income*, 1, 54-61.

Litterman, R., José Scheinkman, L Weiss (1991): "Volatility and the Yield Curve," *Journal of Fixed Income*, 1.

Longstaff, Francis, (1989) "A Nonlinear General Equilibrium Model of the Term Structure of Interest Rates," *Journal of Financial Economics* 23, 195-224.

Menzly, Lior, Tano Santos, and Pietro Veronesi (2004): "Understanding Predictability," *Journal of Political Economy*, 112, 1-47.

Parker, Jonathan (2001): "The Consumption Risk of the Stock Market," *Brookings Papers on Economic Activity*, 2, 279-348.

Pastor, Lubos, and Pietro Veronesi (2003): "Stock Valuation and Learning about Profitability," *Journal of Finance*, 58(5).

Pastor, Lubos, and Pietro Veronesi (2005): "Rational IPO Waves," *Journal of Finance*, 60, 1713 – 1757.

Shreve, Steven E. (2005) *Stochastic Calculus Models for Finance II: Continuous Time Models* (Springer Finance).

Vasicek, O. (1977): "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, 177-188.

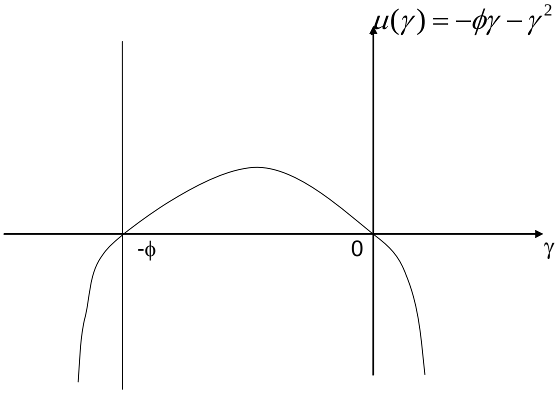

Figure 1: Illustration of the drift $\mu(\gamma) = -\phi\gamma - \gamma^2$ of the growth rate. If $\gamma > -\phi$, the process is stable, i.e. mean reverts to 0. However, if $\gamma < -\phi$, the process is unstable, and diverges away from 0. That is why we impose $\gamma_0 > -\phi$. To make sure that the process remains in $(-\phi, \infty)$, we impose that the volatility goes to 0 fast enough before at some $\underline{\gamma} \geq -\phi$. See Appendix A for details, and Section 7 for the generalization to several factors.

.

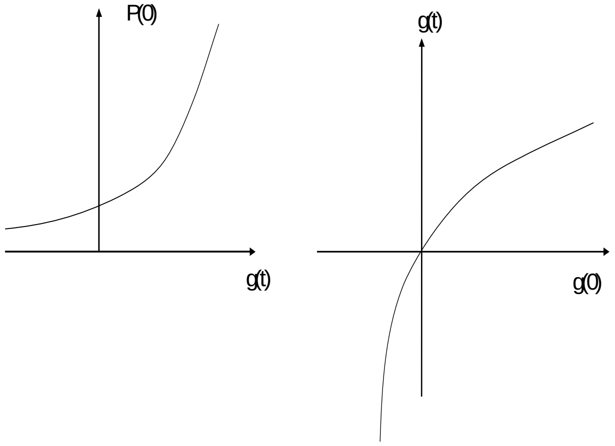

Figure 2: Why the price can be linear in the factor g_0. The price P_0, a sum of $\exp\left(\int_0^T g_t dt\right)$, is a convex function of future growth rates g_t. But, for instance in the deterministic version of the process, future growth rates are a *concave* function of the initial growth rate, $E_0\left[g_t \mid g_0\right]$ is concave in g_0. Hence the price is a composition of a convex function, composed with a concave function the initial growth rate. Hence, its concavity is undertermined. For the LI process, the price P_0 is precisely a linear function of the initial growth rate g_0.

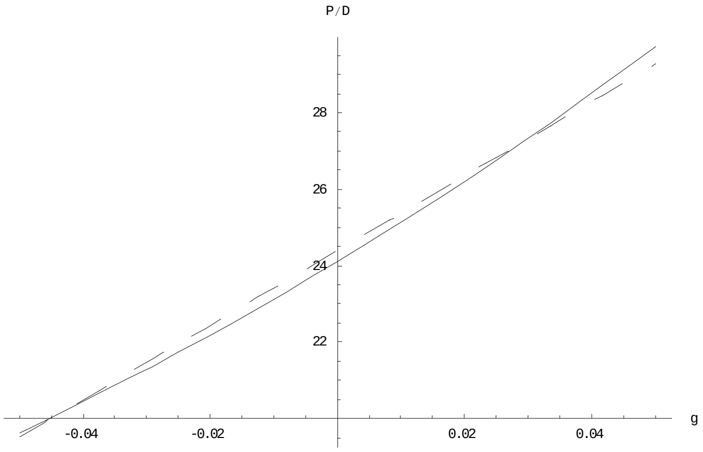

Figure 3: The Figure plots the true value of the P/D ratio of a stock with an Ornstein-Uhlenbeck process (solid line, Eq. 2), and the approximation by a LI process with 1 factor (dashed line, Eq. 68). The annualized values are: $R = 5\%$, $\phi = 15\%$, $\sigma = 4\%$, which corresponds to a stock price volatility of 11% solely caused by changes in g_t. In the range of the Figure, the two curves are within 1.5% of each other.

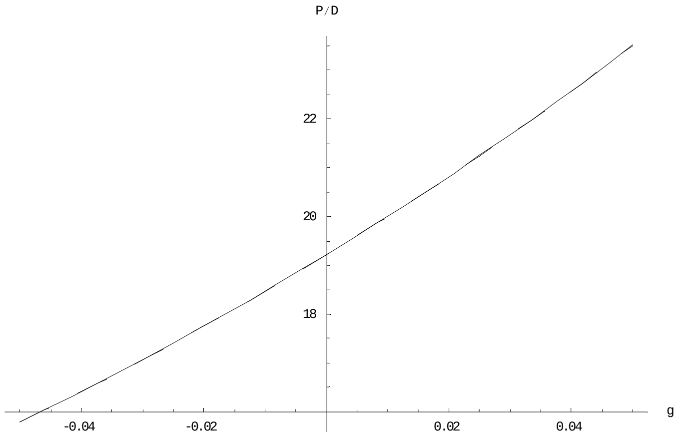

Figure 4: The Figure plots the true value of the P/D ratio of a stock with an Ornstein-Uhlenbeck process (solid line, Eq. 2), and the approximation by a LI process with $n = 5$ factor (solid line), see Eq. 69, truncated at $n = 5$. The annualized values are: $R = 5\%$, $\sigma = 3\%$, $\phi = 15\%$, which corresponds to a stock price volatility of 11% solely caused by changes in g_t. In the range of the Figure, the two curves are within 0.04% of each other.

.