The Risk of Joint Liquidation and Portfolio Choice: Diversity instead of Diversification![*]

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Abstract

Investors in financial markets often have to liquidate when the value of their portfolio has fallen significantly. When many investors have to do so at the same time, prices will typically be depressed. I show that this has important implications for investors' optimal ex-ante portfolio decisions. In particular, it becomes rational for investors to forego diversification benefits. The reason is that the risk of having to liquidate jointly gives investors an incentive to hold diverse portfolios. Investors hence (collectively) invest in all possible portfolio allocations. Some investors even optimally hold completely undiversified portfolios. I also derive a *diversification-irrelevance* result: in equilibrium an investor's pay-off (net of the expected costs of liquidation) is independent of the diversification of his portfolio. There are also consequences for the pricing of assets. An investor has a lower pay-off from investing in assets held by many investors subject to liquidation risk, which is because this increases the risk that he has to liquidate at a time when many others do so as well. Such assets are hence traded at a discount.

JEL classification: G11, G12, G20, G33

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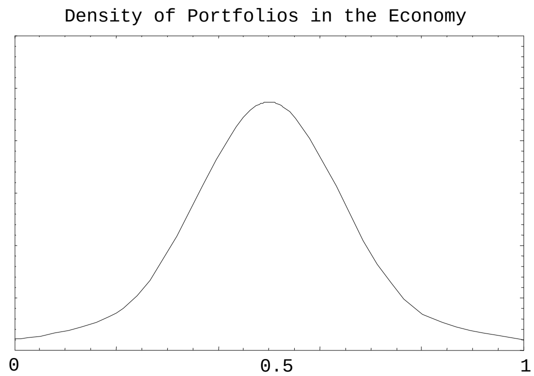
1 Introduction

Investors often have to liquidate following a drop in the value of their portfolios. Since in such situations many investors are typically liquidating at the same time, asset prices are likely to be depressed. For example, several asset-backed securities are currently trading at fire-sale prices because investors were forced to sell them after a decline in their value eroded capital (banks), caused withdrawals (hedge funds), made collateral constraints binding (again hedge funds) or triggered margin calls (traders). Similarly, the current distress in the housing market is a result of liquidations initiated by a drop in house prices which made many highly-mortgaged households essentially insolvent. Another example is the 1987 stock market crash, which has been (at least partly) attributed to automatic trading by institutional investors which stipulated the selling of assets after they had fallen by a certain amount.

This paper shows that the risk of such joint liquidations has important implications for optimal (ex-ante) portfolio allocations, as well as for asset prices. I consider a simple two-asset economy in which investors have to liquidate their portfolio if its value falls below a threshold (this may be, for example, a solvency or a collateral constraint). If an investor's cost of liquidation (arising because he may only be able to sell at low prices) were constant, the standard result obtains that investors diversify fully in order to minimize the risk of liquidation. However, when the liquidation costs are increasing in the number of investors liquidating at the same time, this is no longer the case. The reason is simple. If all investors diversify fully, they hold identical portfolios. Their portfolios will then tend to fall below the thresholds at the same time. This, in turn, would cause investors to liquidate together and produce large liquidation losses. As a consequence of this, investors optimally forego diversification benefits in order to avoid being pooled together in a liquidation. By the same reasoning it is also not optimal for a large amount of investors to jointly invest in *any* given portfolio.[1] Rather, in order to minimize the risk of liquidation with other investors, investors want to hold diverse portfolios in order to be different from each other.

An investor's optimal portfolio allocation in this economy depends on the portfolios of all other investors. This is because these are determining the prices at which he can sell in a liquidation. It turns out that the resulting equilibrium portfolio allocations can be solved for explicitly. A typical distribution of the portfolios looks as follows:

[1] An exception are the fully polarized portfolios, as we discuss in the paper.



The full diversification outcome (investing equally in each asset) is at 0.5, while 0 and 1 refer to the polarized portfolios. The vertical axis shows the density of investors holding a certain portfolio. As can be seen, more diversified portfolios are held by more investors. This is because the risk of liquidation for investors holding such portfolios is lower and hence they accept a higher cost of liquidation by pooling with more investors. We can also see that each possible portfolio is held by at least some investors, which is due to investors' incentives to be as diverse as possible. In particular, there are also some investors who remain fully undiversified.[2]

One can show that a *diversification-irrelevance* result holds in this economy: in equilibrium investors are indifferent about the level of diversification in their portfolios, that is each portfolio gives an investor the same expected return after taking into account the liquidation costs (this, however, does not imply that all assets have the same expected fundamental returns). The intuition for this result is that when all portfolios are played in equilibrium, each portfolio must provide an investor with identical utility since otherwise investors who hold portfolios with lower utility could improve by switching portfolios.

The risk of joint liquidation also has non-standard consequences for asset prices. All else being equal, investors obtain lower pay-offs from investing in assets which are held by many other investors as well because of the higher cost from joint liquidation this entails. Such assets are then traded at lower prices, implying that their (fundamental) returns are higher. For a similar reason, assets which are correlated with the portfolios held by other investors are also traded at a discount.

Why does standard portfolio theory fail to hold in this setting? The textbook case

[2] The polarized outcomes are even played by a "large" number of investors, that is there is a mass point at $\alpha = 0$ and $\alpha = 1$ (not shown in the figure).

assumes that an investors' expected return from investing in an asset is independent of which portfolios other investors are holding. This is not the case here since the portfolio composition of the other investors determines when they have to liquidate and thus also an investor's liquidation costs if he has to liquidate himself. The characteristics of the ownership of assets are thus a determinant of asset prices itself. In particular, assets which are held by many other investors subject to liquidation risk are less valuable. All else being equal, all investors still want to diversify but this is not possible without holding the same assets.

In order to obtain explicit solutions the results are derived in a simple setting where the liquidation constraint and the costs of liquidation are exogenously given. The main insights, however, do not depend on these features. To demonstrate this, I also present a model where forced liquidations arise from insolvency of investors and where the liquidation prices are endogenously determined by the amount of liquidity supplied in financial markets. I show that full diversification remains suboptimal and investors still rationally hold a large variety of different portfolios.

The remainder of this paper is organized as follows. The next section relates to the existing literature. In Section 3 we discuss various channels through which forced liquidations consistent with our model may occur. Section 4 then contains the main analysis of optimal portfolio allocations when there is the risk of joint liquidation. This section also studies the implications for asset prices. Section 5 endogenizes the liquidations. The final section summarizes and makes concluding remarks.

2 Related Literature

The analysis in this paper emphasizes that joint liquidations following low asset values may result in distressed selling. Several important contributions have modelled the presence of such a mechanism. In the industry-equilibrium of Shleifer and Vishny (1992) firms are likely to liquidate at the same time as other firms in the industry. This may cause the liquidation of assets to industry outsiders, who are inefficient users of assets and hence pay lower prices. Shleifer and Vishny also show that firms have an incentive to invest in liquidity in order to benefit from the possibility to purchase assets cheaply. Allen and Gale (1994) introduce the idea of fire-sales in financial markets in the form of so-called *cash-in-the market* pricing: when the supply of liquidity is limited in the short run, a higher number of sellers implicates lower prices per asset. Allen and Gale (2005) then show that low fundamental asset values can cause liquidation at financial intermediaries, resulting in fire-sale prices through cash-in-the-market pricing. Allen and Gale also point out that in any equilibrium fire-sales necessarily have to occur, since otherwise there would be no incentives to provide liquidity ex-ante. Gorton and Huang (2004) and Acharya, Shin and Yorulmazer (2007) explicitly study the incentives for banks to invest in liquidity in order to purchase up assets in a fire-sale. They find that liquidity can be either under- or overprovided.[3] The present paper differs from these papers in that we study the implications of distressed selling for the ex-ante portfolio choice between two risky assets, rather than for the decision to hold liquidity.

There is also an extensive literature which focuses on the financial frictions which may cause distressed prices.[4] Several papers have emphasized that collateral and margin requirements can affect asset liquidity. In Gromb and Vayanos (2002) collateral-constrained arbitrageurs reduce price discrepancies between assets and, in this way, improve liquidity. However, they typically fail to provide the (socially) optimal level of liquidity. Brunnermeier and Pedersen (2007) show that funding liquidity (the ease with which traders can finance themselves in the face of collateral constraints) and market liquidity (the ease with which assets can be sold) can be mutually reinforcing. Acharya and Viswanathan (2009) explicitly endogenize collateral constraints. They show that an endogenous adjustment of the collateral constraints can dampen the effect of liquidity shocks. In Morris and Shin (2004) traders may (simultaneously) hit loss thresholds and have to liquidate their positions. Morris and Shin show that this can cause liquidity black-holes, that is, sudden drops in prices without any apparent fundamental reason. Brunnermeier and Pederson (2005) present a model where forced liquidation at an institution is worsened by predatory traders. These traders initially withdraw liquidity in order to create price swings, from which they can profit later by buying assets.

Several papers have also analyzed the pricing implications of illiquid (or distressed) assets. Acharya and Pedersen (2005) show that the risk of illiquidity is priced into assets ex-ante akin to a capital asset pricing model. Vayanos (2004) develops a dynamic model

[3]Perotti and Suarez (2002) study a related issue. In their model banks benefit from being the "last bank standing", which mitigates their ex-ante risk-taking incentives. In contrast to the beforementioned papers this effect arises through higher gains from being a sole survivor instead of higher losses from joint liquidation.

[4]While the current crisis provides abound examples for asset fire-sales, there is more general evidence that distressed selling can occur in a variety of circumstances, see Asquith, Gertner and Scharfstein (1994), Pulvino (1998), Strömberg (2000), Acharya, Bharath and Srinivasan (2006) and Coval and Stafford (2007).

where fund managers have to liquidate when the value of their funds falls below a certain threshold. He shows that this creates time-varying liquidity premia and other phenomena, such as a "flight to quality". Longstaff (2005) considers a setting where illiquid assets cannot be traded during a certain period and investors are heterogenous. It is shown that illiquidity can substantially reduce the equilibrium price of an asset. In Longstaff (2008) the focus is on the dynamics of distress itself. At the moment when it becomes known that an asset will be in distress, leveraged investors will sell it to less leveraged investors, which causes an increase in the price of safe assets and a reduction in the price of the distressed asset. Later, when uncertainty about the magnitude of distress is resolved, asset prices return to normal. Our analysis adds to these papers in that we show that the liquidity of an asset may crucially depend on the portfolio holdings of all investors in the economy. This makes the liquidity component of asset prices endogenous to assets' ownership characteristics (which, in turn, depend on asset prices itself).

Finally, it is well documented that investors' portfolios are often undiversified (for a survey, see Heaton and Lucas, 2000). There is an extensive literature studying why agents may rationally choose to not diversify. One leading explanation is that investors suffer some direct cost from diversifying, such as for example in the form of transaction costs (e.g., Constantinides, 1986). Another explanation that has been brought forward is that investors may find it optimal not to diversify if they are heterogenous, for example because of different preferences (e.g., Longstaff, 2005) or different background risk (e.g., Heaton and Lucas, 2000). The reason why investors do not diversify in the present paper is not based on either explanation as investors can diversify at no costs and are, moreover, identical. The lack of diversification instead arises because it is costly for investors to invest in similar portfolios. The empirical literature has also emphasized that investors typically hold rather diverse portfolios. Indeed, "the most striking feature of the cross-sectional data is the heterogeneity in portfolio composition..." (Heaton and Lucas, 2000). Interestingly, this diversity arises in our setup without investors themselves being heterogenous.

3 Forced Liquidations

Our analysis is based on the existence of forced liquidations, triggered by low asset values. Specifically, in the baseline model we consider investors (or institutions) whose portfolios have to be liquidated if their value, denoted v, falls below some threshold d. There are several reasons why such liquidations may occur:

- Insolvency: The most straightforward case is the insolvency of an investor or the bankruptcy of a firm. In this case d is simply the amount of debt and when $v < d$ the investor's assets (or the firm) are liquidated (this is the channel we model in Section 5, where we endogenize liquidations). In the case of financial institutions d may refer to the amount of deposits and liquidation may be triggered by a run.[5]

- Insufficient collateral: An investor may have borrowed against the value of its assets. Suppose that a fraction $\beta < 1$ of his assets can be collateralized and that he has borrowed b. Liquidation then occurs if $\beta v < b$, thus $d = b/\beta$.

- Liquidity needs: Investors or firms may use the (intermediate) returns on their portfolio to finance other activities. For example, an investor may use the dividends from his portfolio to meet some liquidity needs l. Suppose that the dividends in a period are a fraction $\beta < 1$ of an asset's total value. The investor may then have to liquidate his portfolio if $\beta v < l$, in which case we have $d = l/\beta$.

- Margin calls: An investor will face a margin call when his margin account falls below a certain threshold, which may force the investor to liquidate his position. Since the margin account reflects his accumulated losses (or gains), this implies a threshold on his portfolio value.

- Withdrawal risk at funds: Following a bad performance of a fund's value, investors may withdraw their money, thus possibly forcing the liquidation of the fund.

- Traders with loss threshold: Traders in financial firms typically have a maximum loss they can incur, and their position will be closed if it exceeds this loss.

- Portfolio insurance: Dynamic trading often involves the selling of an asset when its price declines too much. Similarly, traditional stop-loss orders also stipulate the selling of an asset when its value drops below a certain value.

- Capital requirements: Regulators require financial institutions to hold a certain amount of capital against their risk-weighted capital. A large enough fall in the

[5] In Wagner (2009) liquidations due to runs are endogenized by banks optimally providing liquidity insurance to households through demand deposits. The paper considers the portfolio choice of two banks and shows that it is efficient (from a welfare perspective) for the economy as a whole to be diversified but for individual banks to hold specialized portfolios. The latter is because specialization reduces the likelihood of systemic crises, which induce costs over and above individual bank failures.

value of an institution's assets may make these requirements binding by reducing capital. The institution may then be forced to liquidate assets.

- Loss aversion: Liquidation can also directly arise from investors' preferences. When investors are loss-averse they may liquidate their position if a fall in their portfolio value brings the portfolio close to its initial price.

4 The Model

4.1 Setup

There is a continuum of investors of mass 1. Investors are risk-neutral and are each endowed with one unit of funds. At date 1 they decide on how to divide their funds between two (ex-ante identical) assets, asset X and asset Y. We denote with $\alpha_i \in [0,1]$ the share investor i invests in Y (the share invested in X is hence $1-\alpha_i$). Thus, if $\alpha_i = \frac{1}{2}$ the investor chooses a fully diversified portfolio, while for $\alpha_i = 0$ and $\alpha_i = 1$ the investor holds completely polarized portfolios. The assets mature at date 3. Their returns, denoted x and y, are identically and independently distributed on $[0,\infty)$ with a density $\phi(\cdot)$. We denote their mean with μ.

At date 2, the returns x and y become known. The value of an investor's portfolio is then

$$v_i = (1-\alpha_i)x + \alpha_i y. \tag{1}$$

The liquidation threshold is $d > 0$. Thus, if $v_i < d$, the investor has to liquidate his portfolio. He can only do so at a discount $C(\cdot) \geq 0$ to its fundamental value v_i. $C(\cdot)$ is assumed to be strictly and smoothly increasing in the total mass of assets which are liquidated at date 2 (note that assets are perfect substitutes from date 2 on as there is no longer uncertainty). Since every portfolio has in total one unit of assets, the mass of liquidated assets is identical to the total mass of liquidating investors. We assume that $C(0) = 0$, that is, if the mass of liquidating investors is zero, an individual investor's portfolio can be sold without a loss.

Following liquidation, the investor consumes the proceeds $(v_i - C)$. If there is no liquidation $(v_i \geq d)$, the investor consumes at date 3 the return on his portfolio v_i.

4.2 Optimal Diversification

Since investors are risk-neutral,[6] they maximize the expected returns on their investments. These consist of the expected fundamental value of an investor's portfolio, $E[v_i]$, minus his expected liquidation costs. Since both assets are identically distributed, the expected fundamental value is independent of an investor's portfolio choice: $E[v_i] = \mu$. Hence, maximizing an investor's expected return simply requires minimizing the expected costs from liquidations.

These costs can be derived as follows. An investor has to liquidate his portfolio if its value is below d. From rearranging (1) we have that an investor with portfolio allocation α has to liquidate if $y < \widehat{y}(x)$, where $\widehat{y}(x)$ is given by

$$\widehat{y}(x) = \frac{d}{\alpha} - \frac{1-\alpha}{\alpha}x. \tag{2}$$

$\widehat{y}(x)$ thus denotes the critical return on asset Y for which the investor just avoids liquidation if asset X pays x.

Suppose now that we have $y < \widehat{y}(x)$, that is the investor has to liquidate. The costs this will incur for the investor depend on how many other investors are liquidating at the same time. Consider first $x > y$, that is asset X has a higher return than asset Y. Investors with low α (that is, investors who have not invested much in Y) will then be above the threshold, while investors with high α may be below. One can hence define a critical value $\widehat{\alpha}$ such that all investors with $\alpha > \widehat{\alpha}$ have to liquidate, while all investors with $\alpha \leq \widehat{\alpha}$ do not have to. $\widehat{\alpha}$ is implicitly defined by $v = (1-\widehat{\alpha})x + \widehat{\alpha}y = d$. Rearranging we get

$$\widehat{\alpha}(x, y) = \frac{d-x}{y-x}. \tag{3}$$

We denote in the following with $G(\alpha)$ ($0 \leq G(\alpha) \leq 1$) the mass of investors playing a diversification degree of less than or equal to α. Since all investors with $\alpha > \widehat{\alpha}(x, y)$ liquidate, the mass of liquidated portfolios is hence $1 - G(\widehat{\alpha}(x, y))$. The investor's liquidation costs are consequently $C(1 - G(\widehat{\alpha}(x, y)))$. Consider next $x < y$. Now investors who are more exposed to X may be below the threshold, that is investors with $\alpha < \widehat{\alpha}$. The mass of these investors is given by $G^{-}(\alpha) := \lim_{z \to \alpha^{-}} G(z)$ (note that we may have $G^{-}(\alpha) < G(\alpha)$ if there is point mass at α). The liquidation costs in this case are hence $C(G^{-}(\widehat{\alpha}(x, y)))$.

The different outcomes for an investor with a failure threshold $\widehat{y}(x)$ are summarized in Figure 1. In the area above $\widehat{y}(x)$, there is no liquidation. In area B ($y < \widehat{y}(x)$ and $x > y$)

[6] The diversification benefits arise here because diversification lowers the risk of liquidation. We discuss later the impact of risk-aversion.

there is liquidation with costs of $C(1-G(\widehat{\alpha}(x,y)))$. And in area A ($y<\widehat{y}(x)$ and $x<y$) there is liquidation with costs $C(G^{-}(\widehat{\alpha}(x,y)))$.

The total expected losses from liquidation, denoted with $EL(\alpha)$, can then be found by integrating over all liquidation outcomes:

$$EL(\alpha)=\int_0^d\int_x^{\widehat{y}(x)}\phi(x)\phi(y)C(G^{-}(\widehat{\alpha}(x,y)))dydx+$$

$$\int_0^d\int_0^x\phi(x)\phi(y)C(1-G(\widehat{\alpha}(x,y)))dydx+\int_d^{\widehat{x}(0)}\int_0^{\widehat{y}(x)}\phi(x)\phi(y)C(1-G(\widehat{\alpha}(x,y)))dydx, \tag{4}$$

where $\widehat{x}(0)=\frac{d}{1-\alpha}$ is the x at which $\widehat{y}(x)=0$. The first integral in (4) gives the expected liquidation costs when $x<y$ (area A in Figure 1), while the second and third integral give the expected liquidation costs when $x>y$ (area B). We denote with π_A (π_B) the total probability of liquidations with $x<y$ ($x>y$). These probabilities are given by

$$\pi_A(\alpha) = \int_0^d\int_x^{\widehat{y}(x)}\phi(x)\phi(y)dydx, \tag{5}$$

$$\pi_B(\alpha) = \int_0^d\int_0^x\phi(x)\phi(y)dydx+\int_d^{\widehat{x}(0)}\int_0^{\widehat{y}(x)}\phi(x)\phi(y)dydx. \tag{6}$$

The overall probability of liquidation is then $\pi=\pi_A+\pi_B$. We make the following assumptions on the density ϕ:

(i) The density is smooth and has full support on $[0,\infty)$ (that is we have $\phi(z)>0$ for $z\in[0,\infty)$). From this it follows that $\pi_A'(\alpha)<0$ and $\pi_B'(\alpha)>0$, that is allocating more funds towards Y makes it less likely that an investor has to liquidate if $x<y$ but more likely if $x>y$.

(ii) We have $\pi_A'(\alpha)+\pi_B'(\alpha)<0$ for $\alpha<\frac{1}{2}$ and $\pi_A'(\alpha)+\pi_B'(\alpha)>0$ for $\alpha>\frac{1}{2}$. This guarantees that more diversification (that is, moving towards $\alpha=\frac{1}{2}$ from either side) reduces the overall probability of liquidation π.

(iii) We have $\pi_A''(\alpha)>0$ and $\pi_B''(\alpha)>0$. This ensures that the marginal impact of diversification on the liquidation probability π is declining in the amount of diversification (thus, the benefits from diversification are declining).[7]

We characterize next the equilibrium portfolio allocations in the economy. We do not solve for an investor's individual allocation, as these are not determined. We focus on pure strategies, that is, in an equilibrium each investor chooses an $\alpha_i\in[0,1]$ which minimizes his liquidation costs given the portfolio choices of all other investors.

[7] Assumption (ii) and (iii) hold when the density function is sufficiently flat, and in particular in intervals in which ϕ is constant.

Definition 1 *An equilibrium allocation is given by a mass function $G^*(\alpha)$ ($\alpha \in [0,1]$), such that for all α which are played by at least some investors we have*

$$EL(\alpha) \leq EL(\alpha') \text{ for all } \alpha' \in [0,1]. \tag{7}$$

An equilibrium is thus reached when no investor can lower his expected liquidation costs by switching to another α.

We first show that

Proposition 1 *The full diversification allocation (that is, all investors playing $\alpha = \frac{1}{2}$) is never an equilibrium.*

Proof. *Suppose all investors are fully diversified. Consider an investor who deviates by investing more in Y and less in X (thus, he increases α). The investor will then face more liquidations when $x > y$ since $\pi'_B(\alpha) > 0$ (area B in Figure 1 increases). Since in these situations no other investor liquidates, the costs are $C(0) = 0$. However, when $x < y$ the investor will sometimes avoid liquidations since $\pi'_A(\alpha) < 0$ (area A is reduced). In these situations he previously had to liquidate with all other investors at costs $C(1) > 0$. Thus, his expected liquidation costs are strictly lowered, which implies that full diversification cannot be an equilibrium.* ■

Proposition 1 also holds when $C(0) > 0$. The reason is that if an investor moves (slightly) away from full diversification, the areas of increased and reduced liquidations are identical: $\pi'_A(\frac{1}{2}) + \pi'_B(\frac{1}{2}) = \pi'(\frac{1}{2}) = 0$ (otherwise, full diversification would not minimize the risk of liquidation). Thus, as long as $C(0) < C(1)$ deviating from full diversification will strictly lower an investor's liquidation costs.

The general intuition behind Proposition 1 is that, even though diversification minimizes the risk of liquidation, an investor always wants to avoid holding a portfolio that is held by many other investors as well. This is because he will then tend to have to liquidate at the same time as the other investors have to liquidate, which would incur high costs. The investor could always do better by either increasing or decreasing his diversification degree slightly, which would lower his expected liquidation costs.

This argument does in fact apply to all interior portfolios.[8] One can hence show that investors smoothly distribute themselves over the possible portfolio allocations:

[8] It cannot be applied at $\alpha = 0$ and $\alpha = 1$ because then it may not possible for an investor to deviate in the desired direction. We return to this point later.

Proposition 2 *The equilibrium mass function $G^*(\alpha)$ is smooth on $(0,1)$, that is no interior portfolio is played by a (discrete) mass of investors.*

Proof. *See appendix.* ■

We now solve for the equilibrium allocation. For this we presume that $G^*(\cdot)$ is strictly increasing, that is each α is at least played by some investors (we later show that there cannot be other equilibria). It follows that the expected liquidation costs $EL(\alpha)$ have to be the same for all $\alpha's$. If this were not the case, an investor who plays an α with higher liquidation costs could strictly improve his expected pay-off by moving to an α with lower liquidation costs. It follows that in equilibrium the derivative of $EL'(\alpha)$ has to be zero for all $\alpha \in (0,1)$.

We have for $EL'(\alpha)$

$$\begin{aligned} EL'(\alpha) &= \int_0^d \phi(x)\phi(\widehat{y}(x))C(G^-(\widehat{\alpha}(x,\widehat{y}(x))))\frac{\partial \widehat{y}(x)}{\partial \alpha}dx \\ &+ \int_d^{\widehat{x}(0)} \phi(x)\phi(\widehat{y}(x))C(1-G(\widehat{\alpha}(x,\widehat{y}(x))))\frac{\partial \widehat{y}(x)}{\partial \alpha}dx. \end{aligned} \quad (8)$$

Using that $\widehat{\alpha}(x,\widehat{y}(x)) = \alpha$ (from equations 2 and 3) and substituting in $\pi'_A(\alpha)$ and $\pi'_B(\alpha)$ (which can be obtained from equations 5 and 6) we get the following simple equilibrium condition:

$$\pi'_A(\alpha)C(G^{-*}(\alpha)) + \pi'_B(\alpha)C(1-G^*(\alpha)) = 0. \quad (9)$$

Since $G^*(\cdot)$ is smooth on $(0,1)$ we can alternatively write $\pi'_A(\alpha)C(G^*(\alpha)) + \pi'_B(\alpha)C(1 - G^*(\alpha)) = 0$ for $\alpha \in (0,1)$.

The intuition behind this condition can be readily understood from Figure 1. The figure depicts the impact of an increase in α, which causes a counterclockwise rotation of $\widehat{y}(x)$. When $x < y$ the investor avoids liquidations in area $\triangle A$ and hence lowers his liquidation costs by $\triangle\pi_A C(G^{-*}(\alpha))$. When $x > y$ there are additional liquidations in area $\triangle B$, causing additional liquidation costs of $\triangle\pi_B C(1-G^*(\alpha))$. In an equilibrium both effects have to exactly offset one another, which gives us equation (9). Or in other words: the probabilities associated with $\triangle A$ and $\triangle B$ in Figure 1, weighted with their corresponding liquidation costs, have to be identical.

Note that (9) does not directly define $G^*(0)$ and $G^*(1)$ since we have only shown smoothness on $(0,1)$. $G^*(1)$ can be obtained from the fact that the total mass of all investors has to be 1, hence $G^*(1) = 1$. $G^*(0)$ can be derived from the fact that we must have that $G^*(0) + (1 - G^{+*}(0)) = 1$ (with $G^{+*}(0) = \lim_{z\to 0^+} G^*(z)$). Thus $G^*(0) = G^{+*}(0)$ and $G^*(\alpha)$ is hence continuous at $\alpha = 0$.

Proposition 3 *The mass function $G^*(\alpha)$ defined by*

$$\pi'_A(\alpha)C(G^*(\alpha)) + \pi'_B(\alpha)C(1 - G^*(\alpha)) = 0 \tag{10}$$

for $\alpha \in [0, 1)$ and by $G^*(1) = 1$ is i) unique; ii) strictly increasing; iii) has point masses at $\alpha = 0$ and $\alpha = 1$; iv) is the unique equilibrium mass function.

Proof. *See appendix.* ■

The fact that $G^*(\alpha)$ is strictly increasing is particularly noteworthy. It implies that all portfolio allocations are played by at least some investors. The intuition for this has been already spelled out above: in order to avoid joint liquidations, investors are trying to be as diverse as possible from each other. Hence, they play all possible portfolio outcomes.

Another interesting feature is that there are point masses at both 0 and 1, that is "many" investors choose undiversified outcomes. These investors would in principle like to deviate by either decreasing (at $\alpha = 0$) or increasing (at $\alpha = 1$) their α. However, this is not feasible since they are already fully invested in the asset which they wish to further accumulate.

Note also that in an equilibrium all portfolio strategies incur the same expected liquidation costs. This implies that for an individual investor it does not matter which portfolio allocation he chooses: all allocations give him the same expected return. In a sense, this is a portfolio allocation equivalent of the efficient market hypothesis. The latter says that in equilibrium all securities have the same expected returns, which is because profit maximizing investment behavior causes asset prices to adjust such that returns are equalized. In our setting, it is the profit maximizing portfolio choice of investors which causes expected liquidation costs to be equalized.

4.3 Homogenous Liquidation Costs

Let us assume that the liquidation costs take the form

$$C(m) = c_0 m^{\beta}, \tag{11}$$

where m denotes the mass of selling investors and $c_0, \beta > 0$. We thus have $C(0) = 0$ and $C'(m) > 0$. From (10) we can then derive an explicit formula for $G^*(\alpha)$:

$$G^*(\alpha) = \frac{\pi'_B(\alpha)^{1/\beta}}{(-\pi'_A(\alpha))^{1/\beta} + \pi'_B(\alpha)^{1/\beta}}. \tag{12}$$

Equation (12) shows that the portfolio allocations in the economy are influenced by β, the elasticity of fire-sale prices to the size of aggregate liquidations. In practice, this parameter may relate to the liquidity of financial markets since if the liquidity supply is elastic (that is, markets are liquid) asset prices respond little to aggregate selling pressure and β is low ($\beta \rightarrow 0$ would represent the case of a perfectly elastic supply of liquidity in which liquidation costs are independent of the amount of selling).

Proposition 4 *An increase in β lowers diversification in the economy.*

Proof. *From differentiating $G(\alpha)$ with respect to β one can find that $sign[\frac{\partial G^*(\alpha)}{\partial \beta}] = -sign[\pi'(\alpha)]$. Since $\pi'(\alpha)$, the impact of diversification on the liquidation probability, is negative for $\alpha < \frac{1}{2}$ and positive for $\alpha > \frac{1}{2}$, we thus have $\frac{\partial G^*(\alpha)}{\partial \beta} > 0$ for $\alpha < \frac{1}{2}$ and $\frac{\partial G^*(\alpha)}{\partial \beta} < 0$ for $\alpha > \frac{1}{2}$. Thus, an increase in β shifts probability mass away from $\frac{1}{2}$ towards the polarized outcomes. That is, investor's portfolios become less diversified.* ■

This suggests that in more liquid markets investors can be more diversified. The intuition is straightforward. When there is a lot of liquidity, a joint liquidation of portfolios only causes a small reduction in prices. Hence the costs of joint liquidation are low, allowing investors to pool more on diversified outcomes. For the limit case of $\beta \rightarrow 0$ the liquidation costs C become independent of the number of other investors selling, which is the textbook case. It is straightforward to show from (12) that the portfolio allocations then also converge to the full diversification outcome.

Equation (12) shows, moreover, that the equilibrium portfolio allocation depends on $\pi'_A(\alpha)$ and $\pi'_B(\alpha)$, that is the impact of diversification on the liquidation probabilities. $\pi'_A(\alpha)$ and $\pi'_B(\alpha)$ are determined by the density of the return distribution ϕ and the liquidation threshold d (see equations 5 and 6). Perhaps surprisingly, the impact of d is generally ambiguous and depends on the distribution function. For example, if ϕ is constant in the region affected by a change in α, one can show that the liquidation threshold does not influence the portfolio allocation at all. However, for other distribution functions the effect on diversification may go either way. The reason for this ambiguity is that an increase in d has two counteracting effects on the incentives to diversify. An increase in d increases the risk that an investor has to liquidate. This, on the one hand, increases his incentives to diversify in order to reduce the risk of liquidation. On the other hand, however, it also increases his risk of joint liquidation, which creates an incentive to be more diverse and hence to forego diversification benefits.

Figure 2 depicts $G^*(\alpha)$ and the corresponding density function, denoted $g(\alpha)$, for log-normally distributed asset returns. For this we have assumed linearly increasing costs ($\beta = 1$) and further $d = 1$, $\mu = 2$ and, $\sigma^2 = 1$. We can see that indeed all outcomes are played by at least some investors. Since $G^*(0) > 0$ and $G^{-*}(1) < 1$, we also have that a positive mass of investors is investing in the polarized portfolios. This means that many investors (about 15%) are completely undiversified. Moreover, note that among the investors who diversify ($\alpha \in (0, 1)$), more diversified outcomes are played with a higher density. The reason is that for these outcomes there is a lower risk of liquidation. Hence, investors are prepared to accept higher liquidation costs (once liquidation occurs), implying that they will pool with more other investors. Note also that overall there is a large lack of diversification. For example, the mass of people that is less than "half"-diversified $(G(\frac{1}{4}) + (1 - G(\frac{3}{4}))$ is about 30%.

4.4 Discussion

1. We have implicitly assumed that investors can buy the assets at date 1 at fixed (and identical) costs. Our results also hold when asset prices (at date 1) are endogenous. Suppose that asset X and Y trade at prices p_x and p_y at date 1. The first order condition for α (recall that an increase in α implies more investment in Y and less investment in X) then has to be adjusted by a term $p_y - p_x$ (equation 9). Thus if $p_x = p_y$, the equilibrium is unchanged. Since the equilibrium portfolio allocations in the baseline model are symmetric, they imply an equal total demand for asset X and asset Y. Thus, if the endowments of both assets are also identical, the portfolio allocation of the baseline model will remain an equilibrium with $p_x = p_y$ (the case of asymmetric endowments will be discussed in the next subsection).

2. For illustrative purposes we have considered assets that are identically distributed. This is not important for the results. In fact, Propositions 1-3 also hold if asset returns are asymmetrically distributed. For Proposition 1 one then has to appropriately redefine the full diversification allocation, which no longer will be given by $\alpha = \frac{1}{2}$ (calculations available on request).

3. The non-diversification result is also not specific to the two-assets case. It continues to hold when there are a large number of assets. In this case, however, one has to introduce a common factor (otherwise the market portfolio becomes asymptotically riskless). The intuition is similar to the one of the two-asset setup (Proposition 1). Suppose that all

investors are fully diversified. They then all hold the market portfolio and are hence only subjected to the common factor. If one investor deviates from the market portfolio (for example, if he starts to hold a combination of the market portfolio and an asset), he will in certain situations face more liquidations but there will also be situations where he no longer has to liquidate. In the former case he faces no liquidation costs since he is the only liquidator. By contrast, in the latter case he always saves liquidation costs since he previously had to liquidate with all other investors and hence at high costs. Investors thus always benefit from deviating from the full diversification allocation, which thus cannot be an equilibrium.

4. An interesting question is whether the equilibrium portfolio allocations are efficient. Our baseline model does not permit to analyze efficiency since the purchasers of assets are not considered. In the model presented in Section 5 the purchasers of assets are explicitly modeled. Liquidation costs arise there from *cash-in-the-market* pricing. However, this amounts only to a transfer of wealth from the sellers to the purchasers of assets. There is hence no deadweight cost of liquidation and all portfolio allocations are efficient.

5. Investors are risk-neutral in our analysis and the rationale for diversification arises because diversification lowers the likelihood of liquidation. Introducing risk-aversion gives rise to two new effects of diversification. On the one hand, higher diversification reduces portfolio variance in the absence of liquidation. This makes diversification more attractive. On the other hand, more diversified portfolios entail higher liquidation costs once liquidation occurs (as discussed in Section 4.3). These are outcomes with very low pay-offs, which become more costly (in utility terms) when investors are risk-averse. This makes diversification less attractive.[9]

The non-diversification result, however, continues to hold. This can be appreciated from the fact that close to full diversification the impact of diversification on the variance of a portfolio is zero (otherwise diversification would not be variance minimizing). Hence, for an investor who deviates from a full diversification allocation, the first effect considered above vanishes. Thus, only the second effect remains. Relative to the effects under risk-neutrality (see proof of Proposition 1), it becomes then even more attractive to deviate from a full diversification allocation. Thus full diversification still cannot be an equilibrium.

[9]For CARA-utility one can solve for the equilibrium allocations, which reveal that the impact of risk-aversion on the amount of diversification is ambiguous (calculations available on request).

4.5 Implications for Asset Prices

When joint liquidation is costly, an investor's return is no longer independent of the portfolios held by other investors. This has interesting implications for asset pricing. Let us ignore for the moment the possibility for diversification by assuming that investors either invest in asset X or in asset Y. Denote with G_x the mass of investors who invest in asset X and with G_y the mass invested in Y. Furthermore, denote the likelihood that an asset's return falls below the threshold d with π_d.

Since investors are risk-neutral, the price of an asset is simply its expected pay-off from holding it. In our setting, however, this is not equal to its fundamental pay-off, since there are also the liquidation costs. The latter are now determined as follows:

- with probability $(1-\pi_d)\pi_d$ we have $x < d$ and $y \geq d$: all investors in X then have to liquidate and face liquidation costs of $C(G_x)$;
- with probability $(1-\pi_d)\pi_d$ we have $x \geq d$ and $y < d$: all investors in Y have to liquidate and face liquidation costs of $C(G_y)$;
- with probability π_d^2 we have $x, y < d$: all investors have to liquidate and the liquidation costs are $C(G_x + G_y) = C(1)$.

The overall expected return from investing in either asset (and hence its price) is thus

$$p_x = \mu - (1-\pi_d)\pi_d C(G_x) - \pi_d^2 C(1), \tag{13}$$

$$p_y = \mu - (1-\pi_d)\pi_d C(G_y) - \pi_d^2 C(1) \tag{14}$$

We have $\partial p_x/\partial G_x, \partial p_y/\partial G_y < 0$, thus the price of an asset depends negatively on the amount of investors holding it.[10] The intuition for this result is straightforward. An asset which is held by many other investors is less attractive because in the case an investor has to liquidate his position, many other investors will be selling as well and prices will hence be severely depressed. Investors will thus require a higher (fundamental) return on such an asset and its price will be lower.

Our analysis suggests that, ceteris paribus, assets held by many investors are effectively less liquid in liquidations. The reason for this apparent paradox is that liquidity is a two-sided phenomenon: it requires a balance between the demand and the supply of an asset.

[10] Note that for the textbook case of constant C we obtain $p_x = p_y = \mu - \pi_d C$ and prices are independent of the ownership of assets.

And when many investors hold an asset at the same time, a drop in its value may result in a large (and sudden) increase in the supply of the asset through forced liquidations. Note that this notion of illiquidity is one of illiquidity in stress times. In normal times such an asset may be very liquid, precisely because many investors are holding it.

We have thus far assumed that the investors' portfolios are exogenously given and completely polarized. The following proposition shows that the results also hold for the equilibrium portfolio allocations:

Proposition 5 *Assets which in equilibrium are held by more investors trade at lower prices.*

Proof. *Denote the endowments of asset X and Y in the economy with S_x and S_y ($S_x + S_y = 1$). When choosing their portfolios, individual investors take asset prices p_x and p_y as given. Thus, the new condition for equilibrium diversification is (recalling that an increase in α refers to more investment in Y and less in X):*

$$p_y - p_x + \pi'_A(\alpha)C(G^*(\alpha)) + \pi'_B(\alpha)C(1 - G^*(\alpha)) = 0. \tag{15}$$

Denoting the price differential between the assets with $\triangle p$ ($\triangle p = p_y - p_y$), we get from totally differentiating (15) with respect to $G^*(\alpha)$

$$\frac{d \triangle p}{dG^*(\alpha)} + \pi'_A(\alpha)C'G(^*(\alpha)) - \pi'_B(\alpha)C'(1 - G^*(\alpha)) = 0. \tag{16}$$

Since $\pi'_A(\alpha) < 0$ and $\pi'_B(\alpha) > 0$, it follows that $\frac{d\triangle p}{dG^*(\alpha)} > 0$. Market clearing requires that

$$S_x = \int_0^1 (1 - \alpha)G'(\alpha)d\alpha \qquad S_y = \int_0^1 \alpha G'(\alpha)d\alpha. \tag{17}$$

Using the method of partial integration we obtain

$$S_x = \int_0^1 G(\alpha)d\alpha \qquad S_y = 1 - \int_0^1 G(\alpha)d\alpha. \tag{18}$$

It follows that an increase in S_x has to increase $G(\alpha)$ for at least some α. Since $\frac{\triangle p}{dG^*(\alpha)} > 0$ for all α, it follows that $\triangle p$ has to increase as well. Thus, if asset X is held in equilibrium by more investors, its price relative to asset Y decreases. ■

One can also show that an asset that is more correlated with an asset held by many other investors has a lower price as well. To this end suppose that all investors are holding asset X. Furthermore, assume that with probability ρ asset returns are perfectly correlated, while with probability $1 - \rho$ they remain independent. The liquidation outcomes for an investor who (hypothetically) holds asset Y are then as follows. When returns are correlated, the investor has to liquidate with probability π_d, in which case he has to liquidate with all other investors and incurs costs of $C(1)$. When the returns are independent, the probability that an investor has to liquidate alone is $(1-\pi_d)\pi_d$. In this case he incurs zero liquidation costs. The probability that all investors have to liquidate in this case is π_d^2 and liquidation costs are then $C(1)$.

The price of asset Y is hence given by

$$p_y = \mu - (1-\rho)\pi_d^2 C(1) - \rho\pi_d C(1). \tag{19}$$

It follows that $\partial p_y / \partial\rho < 0$, that is a higher correlation with the asset held by the other investors reduces an asset's price. This is akin to the CAPM, where an asset's required return increases with its correlation with the market portfolio. The difference is that in the CAPM world it is the correlation with the "average" security in the economy which matters, while in our setting it is the correlation with the security held by the "average" investor.

5 Endogenous Liquidation

The baseline model allowed us to derive clear answers to a relatively complex problem (an investor's diversification choice in principle depends on the liquidation costs in all possible liquidation states, which in turn each depend on the portfolio allocations of all other investors in the economy). However, it is stylized in several important respects. In particular it does not explain why portfolios have to be liquidated at date 2. Moreover, the liquidation cost function is exogenously given and the purchasers of assets in a liquidation and their incentives to provide liquidity were not considered.

In this section we modify the basic model in order to deal with these issues. We consider investors who want to raise debt at date 1 in order to (partially) finance their portfolio investment. This may cause insolvency at date 2 when the value of their portfolio is low, resulting in liquidation of the portfolio. The cost of liquidation are determined by *cash-in-the-market pricing* (e.g., Allen and Gale 1994): when the available liquidity is insufficient to purchase all liquidated portfolios at their true value, prices will be below fundamental values. We show that in equilibrium portfolios are always liquidated at a loss in order to provide incentives for holding liquidity. The supply of liquidity is determined by the decision of agents at date 1 whether to invest or to provide liquidity. This decision trades off the higher fundamental returns from investing (net of expected liquidation costs) with the benefits of having the possibility to acquire assets cheaply at date 2 by holding liquidity. In equilibrium, both holding liquidity and investing of course yield the same expected return.

The main results carry through. In particular, the full diversification outcome remains inefficient (Proposition 1) and the density is smooth for all interior portfolios (Proposition 2), that is many diverse portfolios are invested in. Under appropriate parameter constellations there is also density everywhere, that is all portfolios are played (when this is not the case, the equilibrium may not exist).

More specifically, we modify the baseline model as follows. There is now a continuum of agents, which are endowed with $1 - d$ units of capital each. At date 1 an agent can decide whether to undertake investment activities (that is, to become an "investor") or not. Investing funds (up to a size of 1 as in the baseline model) incurs private costs of $e > 0$, which can be interpreted as the time and resources spend on investment activities. Since the endowment is only $1 - d$ (< 1), an investor needs d units of external financing in order to invest the maximum amount (we discuss later why the investor may find it optimal to do so). We assume that he can raise these funds by borrowing from a widely dispersed number of creditors.[11] As in the baseline model, the investor also decides on his α by dividing his funds between the assets X and Y. We assume that investment is a positive NPV activity: $\mu - e > 1$. If an agent decides not to become an investor he can either lend his funds to investors (a "creditor") or simply hold his funds as liquidity (a "liquidity provider").

At date 2, two situations can arise for an investor. If $v \geq d$ (we assume that interest is not accrued at the intermediate date) he is solvent. He can then continue and will ordinarily repay his debt at date 3. If $v < d$, the investor is insolvent. Because creditors are dispersed, neither can the investor renegotiate his debt nor can creditors coordinate on continuing the assets themselves. The assets are hence liquidated and purchased by the liquidity providers, with the proceeds going to the creditors.

We next derive the liquidation costs at date 2. The supply of liquidity at date 2 is given by the total funds of all liquidity providers. We denote the mass of agents who have chosen to be an investor with β. The liquidity supply can then be written as the funds of all non-investors $(1 - \beta)(1 - d)$ minus the funds of all creditors βd:

$$L := (1 - \beta)(1 - d) - \beta d = 1 - d - \beta. \tag{20}$$

[11] Following Hart and Moore (1994), dispersed debt may be necessary due to borrower moral hazard (see also Berglöf and von Thadden (1994), Bolton and Scharfstein (1996) or Dewatripont and Tirole (1994)).

Suppose first that $x < y$. Then all portfolios with $\alpha < \widehat{\alpha}(x,y)$ have to be liquidated. The total (fundamental) value of all liquidations is then $\lim_{x\to\alpha^-}\int_0^x g(z)v(z)dz$, where g is the density function corresponding to G and v is defined as in equation (1). When this is less than or equal to L, the available liquidity is sufficient to purchase all portfolios at their fundamental value. Investors do not then face any liquidation costs (we show later that such situations do not arise in equilibrium). However, when $\lim_{x\to\alpha^-}\int_0^x g(z)v(z)dz > L$, this is no longer the case and there is *cash-in-the-market pricing.* All available liquidity is then used to purchase assets. The price per liquidated portfolio is hence (assuming that all liquidated portfolios are sold at the same price) simply determined by the ratio of the liquidity supply to the mass of liquidated portfolios

$$p = \frac{L}{G^{-}(\alpha)}. \tag{21}$$

The price thus increases with the liquidity supply L and falls with the mass of liquidating investors. Similarly, there is *cash-in-the-market* pricing if $\int_\alpha^1 g(z)v(z)dz > L$ and the resulting price is

$$p = \frac{L}{\beta - G(\alpha)}. \tag{22}$$

The expected returns of the agents are as follows. Consider first an investor. We denote the required return of creditors (which will be determined in equilibrium) by i. Given that investors borrowed d, their expected repayment to creditors is hence $(1+i)d$ (note that i will be smaller than the nominal interest rate when there are also states of default). An investor's overall expected return is hence given by the expected fundamental return on his portfolio, μ, minus the costs of debt, $(1+i)d$, minus private costs, e, minus expected liquidation costs, $EL(\alpha)$. The latter are as in the baseline model except that the liquidation discount is now given by $v - p(G^{-}(\widehat{\alpha}(x,y)))$ when $x < y$ and by $v - p(\beta - G(\widehat{\alpha}(x,y)))$ when $x > y$. We thus have that an investor's expected return, denoted u_I, is

$$\begin{aligned} u_I &= \mu - (1+i)d - e - \int_0^d \int_x^{\widehat{y}(x)} \phi(x)\phi(y)(v(x,y) - p(G^{-}(\widehat{\alpha}(x,y))))dydx \\ &\quad - \int_0^d \int_0^x \phi(x)\phi(y)(v(x,y) - p(\beta - G(\widehat{\alpha}(x,y))))dydx \\ &\quad - \int_d^{\widehat{x}(0)} \int_0^{\widehat{y}(x)} \phi(x)\phi(y)(v(x,y) - p(\beta - G(\widehat{\alpha}(x,y))))dydx. \end{aligned} \tag{23}$$

We assume that the cost of debt fully reflects an investors' portfolio choice. Therefore, the investor takes the expected payment to debtors $(1+i)d$ as given when he decides on his portfolio. This implies that his objective is to minimize the expected liquidation costs $EL(\alpha)$ (the sum of the integrals in equation 23) as in the baseline model.

Creditors simply get on average the required return $1 + i$ on their endowment

$$u_D = (1 - d)(1 + i). \tag{24}$$

Liquidity providers obtain a return of 1 on their liquidity holdings in the absence of liquidations (liquidity is then simply stored until the final period and consumed). When there is liquidation, however, the return may be higher. One unit of liquidity buys $1/p$ of a portfolio, which has an expected fundamental value of $\widetilde{v}_{\alpha<\widehat{\alpha}} := \lim_{x\to\alpha^-} \int_0^x g(z)v(z)dz/G^-(\alpha)$ when $x < y$ and $\widetilde{v}_{\alpha>\widehat{\alpha}} := \int_\alpha^1 g(z)v(z)dz/(\beta - G(\alpha))$ when $x > y$. Hence there is an excess return if $\widetilde{v}/p - 1 > 0$ and it can be easily verified that this is indeed the case whenever there is *cash-in-the-market* pricing. The expected return for a liquidity provider is thus

$$\begin{aligned} u_L = (1-d)(1 &+ \int_0^d \int_x^\infty \phi(x)\phi(y)(\frac{\widetilde{v}_{\alpha<\widehat{\alpha}}}{p(G^-(\widehat{\alpha}(x,y))} - 1)dydx \\ &+ \int_0^d \int_0^x \phi(x)\phi(y)(\frac{\widetilde{v}_{\alpha>\widehat{\alpha}}}{p(\beta - G(\widehat{\alpha}(x,y))} - 1)dydx \\ &+ \int_d^\infty \int_0^d \phi(x)\phi(y)(\frac{\widetilde{v}_{\alpha>\widehat{\alpha}}}{p(\beta - G(\widehat{\alpha}(x,y))} - 1)dydx). \end{aligned} \tag{25}$$

We denote in the following the expected liquidation benefits from a unit of liquidity by EB (sum of the three integrals in equation 25).

An *equilibrium allocation* can be characterized by a mass of investors β^*, a return on debt i^* and a mass function $G^*(\alpha)$ on $[0, 1]$ such that

(i) the expected returns for all agents are identical: $u_I = u_L = u_D$;

(ii) for each α that is played by at least one investor we have that $u_I(\alpha) \geq u_I(\alpha')$ for all $\alpha' \in [0, 1]$.

Note that β alone determines the mass of each class of agents. In particular, if there are β investors we have that total debt is βd. Thus, there are $\frac{\beta d}{1-d}$ creditors. The remaining agents $(1 - \beta - \frac{\beta d}{1-d})$ are then liquidity providers. Note also that any liquidity provision is in fact (socially) inefficient since investment has a positive NPV and hence all funds in the economy should be invested.

We first take β and i as given and show that an equilibrium allocation $G^*(\alpha)$ has similar properties as in the baseline model. In particular:

1. $G^*(\alpha)$ is smooth for all interior portfolios (from this follows, in particular, that full diversification is still not an equilibrium).

2. There are mass points at $\alpha = 0$ and $\alpha = 1$.

3. For appropriate parameter constellations $G^*(\alpha)$ is strictly increasing (when this is not the case the equilibrium may not exist).

Consider first the impact of an increase in α on the expected liquidation losses $EL\left(\alpha\right)$ (the sum of integrals in equation 23)

$$
\begin{aligned}
EL'(\alpha) &= \pi_A'(\alpha)(d - p(G^-(\alpha)) + \pi_B'(d - p(\beta - G(\alpha)) \\
&\quad + \int_0^{\widehat{x}(0)} \int_0^{\widehat{y}(x)} \phi(x)\phi(y)(y-x)dydx. \qquad (26)
\end{aligned}
$$

The first and the second expression in (26) are identical to the baseline model (see equation 10) for liquidation costs $C = d - p$. These expressions arise because a change in α changes the areas in which portfolios have to be liquidated. The reason why the (marginal) liquidation costs are $d - p$ is that d is the (fundamental) value of a marginal portfolio (that is, a portfolio that just has to be liquidated) and p is the price at which it can be sold (we show later that that in equilibrium the liquidation costs are always positive, that is $d > p$).

The third expression in (26) arises because diversification now not only changes the liquidation area but also the liquidation costs within the liquidation area. The reason is that diversification alters the value of an investor's portfolio v in liquidation and hence also the liquidation costs. This effect was not present in baseline model since there the liquidation costs were assumed to be independent of the value of the liquidated portfolio. Using that X and Y are identically distributed we can rewrite the third expression to yield

$$
E(\alpha) := \int_0^d \int_{\frac{d}{1-\alpha} - \frac{\alpha}{1-\alpha}x}^{\widehat{y}(x)} \phi(x)\phi(y)(y-x)dydx. \qquad (27)
$$

For $\alpha < \frac{1}{2}$ we have that the upper integration bound is strictly larger than the lower. Since $y - x > 0$ (note that $y - x = \frac{\partial v}{\partial \alpha}$ is the marginal impact of diversification on the portfolio value) within the integration bounds, it follows then that $E(\alpha) > 0$. Conversely, for $\alpha > \frac{1}{2}$ we have $E(\alpha) < 0$. The reason why the costs increase is that diversification raises the average value of an investor's portfolio in liquidation and thus also the liquidation costs (note that such an effect would also be present if the liquidation costs were proportional to the value of a portfolio).

We show now that $p < d$, that is there is a liquidation cost for the marginal liquidation portfolio (that is, a portfolio with $\alpha = \widehat{\alpha}(x, y)$). To this end suppose first that the marginal portfolio is never sold at a loss: $p \geq d$ for all liquidation outcomes. Since the marginal portfolio has the highest value among all liquidated portfolios this implies that all other portfolios are sold without a loss as well. This implies in turn that liquidity providers never benefit from liquidity at date 2. Hence we have $u_L \leq 1-d$. Since in equilibrium $u_D = u_L$ it follows that $i \leq 0$ and hence $u_I \geq \mu - d - e > 1-d$. Thus we have $u_I > u_L$, which contradicts the equilibrium condition $u_I = u_L$. The intuition for this result is simple. If portfolios are never sold below value all agents would want to invest and no agent would want to hold liquidity. But then no liquidity is supplied at date 2 and hence portfolios could never be sold at their true value.

We thus already know that at least sometimes the marginal portfolio is sold at a loss. We show next that it is always sold at a loss. For this suppose to the contrary that there are some x, y for which it is not sold at a loss ($p \geq d$). Without loss of generalization suppose that this happens when $x < y$ and $\widehat{\alpha}(x,y) < \frac{1}{2}$. Since the amount of portfolio liquidations $G^-(\widehat{\alpha})$ must be (weakly) increasing in $\widehat{\alpha}$ (and hence prices decreasing in $\widehat{\alpha}$), there is thus a critical $\overline{\alpha}$, such that if $\widehat{\alpha}(x,y) > \overline{\alpha}$ there is selling at a loss but when $\widehat{\alpha}(x,y) \leq \overline{\alpha}$ there is not (this $\overline{\alpha}$ exists since we have shown that at least sometimes we have $p < d$). For this $\overline{\alpha}$ we have that there is also selling at a loss when $x > y$ because the amount of liquidations is then even higher ($\beta - G(\overline{\alpha}) > G^-(\overline{\alpha})$ since $\overline{\alpha} < \frac{1}{2}$ and hence $G(\overline{\alpha}) < \frac{\beta}{2}$).

We also know that at least some investors are playing $\overline{\alpha}$ (this is because at $\overline{\alpha}$ total portfolio liquidations have to be increasing in order for $\overline{\alpha}$ to be the critical value). Suppose that such an investor decreases his α a bit (that is, shifts more into asset X). He thus fails more when $x < y$ but this is not costly since there are no liquidation costs for $\alpha < \overline{\alpha}$. However, in the area where he fails less (arising when $x > y$) he previously incurred positive liquidation costs. Thus, the change in the liquidation areas lowers his liquidation costs (first and second term in equation 26). Moreover, the costs of liquidation within the liquidation area also fall (since we have $E(\alpha) > 0$ for $\alpha < \frac{1}{2}$, equation 27). Thus, this cannot be an equilibrium. It follows that the marginal portfolio is always sold at a loss. From this follows in turn that there is always cash-in-the-market pricing when there are liquidations. We can thus substitute p in the expressions for the return of investors and liquidity holders with their *cash-in-the market* price (equations 23 and 25).

The fact that there is always *cash-in-the-market* pricing has direct implications for the equilibrium G^*. When $\widehat{\alpha}(x,y) = 1$ and $x > y$, $\beta - G^-(1)$ portfolios are liquidated. Since there is always a shortage of liquidity (as just shown), we must have in such a situation that $d(\beta - G^-(1)) > L$. Thus, $G^-(1) < \beta$. Similarly, when $\widehat{\alpha}(x,y) = 0$ and $x < y$, $G(0)$ portfolios are liquidated. We thus have $dG(0) > L$ and hence $G(0) > 0$. It follows that there are mass points at $\alpha = 0$ and $\alpha = 1$ (Property 2. above).

We show next that G^* is smooth for all interior portfolios (Property 1. above). The proof is analogous to the one for Proposition 2. Suppose that $G(\alpha)$ is not smooth on $(0,1)$. There exists then an α_0 at which there is point mass. Without loss of generality suppose that $G^{-}(\alpha_0) < G(\alpha_0)$ (that is, the jump is at α_0). In order for G to be an equilibrium mass function an investor who plays α_0 (such an investor exists since there is mass at α_0) should not be able to reduce his expected liquidation costs by either lowering or increasing α slightly. The condition that the investor cannot reduce his liquidation costs by lowering α is (from equation 26)

$$\pi_A'(d - \frac{L}{G^{-}(\alpha_0)}) + \pi_B'(d - \frac{L}{(\beta - G^{-}(\alpha_0))}) + \int_0^{\widehat{x}(0)} \int_x^{\widehat{y}(x)} \phi(x)\phi(y)(y-x)dydx \leq 0. \quad (28)$$

Similarly, the condition that an investor cannot reduce the costs by increasing α is

$$\pi_A'(d - \frac{L}{G(\alpha_0)}) + \pi_B'(d - \frac{L}{(\beta - G(\alpha_0))}) + \int_0^{\widehat{x}(0)} \int_x^{\widehat{y}(x)} \phi(x)\phi(y)(y-x)dydx \geq 0. \quad (29)$$

Since $\pi_A' < 0$, $\pi_B' > 0$ and $G^{-}(\alpha_0) < G(\alpha_0)$, we have $\pi_A'(d - \frac{L}{G^{-}(\alpha_0)}) > \pi_A'(d - \frac{L}{G(\alpha_0)})$ and $\pi_B'(d - \frac{L}{(\beta - G^{-}(\alpha_0))}) > \pi_B'(d - \frac{L}{(\beta - G(\alpha_0))})$, and hence (28) and (29) form a contradiction.

Finally we show that there are parameter constellations for which $G(\alpha)$ is (strictly) increasing on the entire interval (Property 3.). If $G(\alpha)$ is increasing the equilibrium G^* is given by the condition $EL'(\alpha) = 0$. Totally differentiating with respect to α (using equation 26) gives

$$\begin{aligned} &\pi_A''(\alpha)(d - \frac{L}{G^*(\alpha)}) + \pi_B''(\alpha)(d - \frac{L}{\beta - G^*(\alpha)}) \\ +&\pi_A' \frac{L}{(G^*(\alpha))^2} G^{*\prime}(\alpha) - \pi_B' \frac{L}{(\beta - G^*(\alpha))^2} G^{*\prime}(\alpha) + E'(\alpha) = 0. \end{aligned} \quad (30)$$

Rearranging for $G^{*\prime}$ we obtain

$$G^{*\prime}(\alpha) = \frac{\pi_A''(\alpha)(d - \frac{L}{G^*(\alpha)}) + \pi_B''(\alpha)(d - \frac{L}{\beta - G^*(\alpha)}) + E'(\alpha)}{\pi_B' \frac{L}{(\beta - G^*(\alpha))^2} - \pi_A' \frac{L}{(G^*(\alpha))^2}}. \quad (31)$$

We have $\pi_B' > 0$ and $\pi_A' < 0$, hence the denominator is positive. In the nominator we have $\pi_A''(d - \frac{L}{G^*(\alpha)}), \pi_B''(d - \frac{L}{\beta - G^*(\alpha)}) > 0$ and $E'(\alpha) < 0$ (from differentiating equation 27. It can be shown (calculations available on request) that there are parameter constellations for which $\pi_A''(\alpha)(d - \frac{L}{G^*(\alpha)}) + E'(\alpha) > 0$ (intuitively, this happens if L is small which can be achieved in equilibrium by making the portfolio returns μ large). Then the nominator is also positive, from which it follows that $G^*(\alpha)$ is strictly increasing.

This concludes our analysis of the equilibrium portfolio allocation (for which we have assumed β and i given). In what follows we characterize the equilibrium levels of β and

i (however, they cannot be explicitly solved for). Note that $G^*(\alpha)$ depends on β but not on i since $EL'(\alpha)$ is only a function of β. For each β we thus obtain a $G^*(\alpha)$, which in turn defines the expected liquidation losses EL and the expected liquidity benefits EB (the integrals in equation 25). We have $EL'(\beta) > 0$ and $EB'(\beta) > 0$ since a higher share of investors means more liquidated portfolios and less available liquidity. This implies lower prices in liquidations which induces higher losses for investors but benefits liquidity providers.

In equilibrium we have that liquidity providers' and investors' expected returns are the same ($u_L = u_I$). This can be written as

$$(1-d)(1+EB(\beta)) = \mu - (1+i)d - e - EL(\beta). \tag{32}$$

Furthermore, solving the condition that the expected returns on debt and liquidity are the same ($u_L = u_D$) for i gives

$$i = EB(\beta). \tag{33}$$

Using this to substitute i in (32) and rearranging gives

$$1 + EB(\beta) = \mu - e - EL(\beta). \tag{34}$$

Since $EB'(\beta) > 0$ and $EL'(\beta) > 0$ as discussed above, this defines us a unique β^*. From (33) we can then also get the equilibrium i^*.

In our analysis we have presumed that an investor finds it optimal to invest one unit of funds, the maximum possible amount. What happens if an investor invests less? This would allow him to raise less debt. Or, alternatively, he may leave debt unchanged and hold liquidity. Consider the first possibility. Suppose that an investor with arbitrary α deviates from an equilibrium in which all investors are fully invested by only investing $1-\gamma < 1$ units of funds. The amount of debt he has to raise is then $d-\gamma$. Since his portfolio value is now $(1-\gamma)((1-\alpha)x + \alpha y)$, he will have to liquidate at date 2 if

$$(1-\gamma)((1-\alpha)x + \alpha y) < d - \gamma. \tag{35}$$

Rearranging, we find that the critical return $\widehat{y}(x)$ is

$$\widehat{y}(x) = \frac{d-\gamma}{\alpha(1-\gamma)} - \frac{1-\alpha}{\alpha}x, \tag{36}$$

and is falling in γ. Thus, raising less debt and investing less lowers the risk of liquidation. Lower investment also reduces an investor's portfolio value in liquidation, which in turn reduces his liquidation costs (similar to the third effect induced by a change in α discussed above, see equation 26). Thus, the liquidation costs are lowered. The investor has to contrast the benefits arising from this with the loss of the return from investment. The (marginal) excess return from investment (including the cost of debt) is $\mu - (1 + i)$ (e does not appear here since these are fixed costs). Thus, if μ is large relative to the saved liquidation costs, the investor will not find it optimal to invest less than the full amount.

In fact, one can show that there are constellations for which this is the case. To see this suppose that we have an equilibrium where all investors are invested fully. Consider now that we shift density from low non-failure states to high non-failure states (non-failure states are states for which $x, y > d$). In this way we can make μ arbitrarily high. At the same time we increase e such that $\mu - e$ stays constant. This will not change the equilibrium since neither the expected liquidation losses $EL(\alpha)$ nor the excess returns $\mu - e$ are affected (equation 23 is unchanged). For the same reason it will also not change the required return on debt i. It also follows that the marginal benefits from investing less by raising less debt, which arise from a lower likelihood of liquidation and a lower cost of liquidation, remain the same. However, the marginal costs of doing so ($\mu - (1 + i)$) become arbitrarily large. Thus, there are parameter constellations for which the investor indeed finds it optimal to be fully invested. In a similar fashion one can also rule out that an investor wants to hold liquidity. An additional feature in this case is that liquidity can also be used to purchase portfolios from other investors if they have to liquidate

6 Conclusion

This paper analyzes portfolio choices and asset prices when investors have to liquidate following a low return of their portfolios. The main idea is that when many investors have to do so at the same time, asset prices will be depressed and liquidation will become costly. Rational investors should anticipate this and hence choose a portfolio which reflects the risk of joint liquidation.

We have shown that the prescription of standard portfolio theory, which stipulates that all investors should hold a fully diversified portfolio, does not then apply. This is even though investors are completely homogenous and there is no cost of diversifying per se. The reason for this is that there is a trade-off between diversification and diversity. If all investors are fully diversified, their risk of liquidation is minimized but they then also hold identical portfolios. Hence whenever there is liquidation, investors will all tend to liquidate together and asset prices will be severely depressed. To avoid the resulting substantial liquidation losses, investors do not find it optimal to jointly invest in the diversification outcome. Instead, they want to hold diverse portfolios in order to reduce the risk of jointly liquidating.

An important insight from this reasoning is that an investor's desired portfolio allocation depends crucially on which portfolios the other investors in the economy are holding. In particular, investors value an asset which is held by many other investors, or is correlated with an asset held by many others, less. Such assets then command a lower price in equilibrium. Asset prices are hence, besides their fundamentals, determined by their ownership.

The analysis raises several interesting questions for empirical and theoretical work. Besides providing a general rationale for underdiversification, the setup may potentially help to explain the wide dispersion in the degree of diversification in the portfolios of otherwise similar investors. The analysis also has implications for the portfolio allocations of investors which face a high risk of joint liquidation (such as, for example, hedge funds) relative to those of investors with modest liquidation risk (such as unleveraged private investors). Furthermore, there are consequences for the behavior of the prices of assets that differ with respect to their ownership characteristics.

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Appendix (Proofs)

Proof of Proposition 2. Suppose that $G^*(\alpha)$ is not smooth on $(0,1)$. There exists then an α_0 at which there is a mass point. Without loss of generality suppose that $G^-(\alpha_0) < G(\alpha_0)$ (that is the jump is at α_0 and not to its right). If this is an equilibrium, then an investor who plays α_0 (such investors exist since there is point mass at α_0) should not be able to lower his expected liquidation costs by either lowering or increasing α slightly. From differentiating $EL(\alpha)$ (equation 4) with respect to α we get that the marginal impact of α on the expected liquidation costs is

$$EL'(\alpha) = \pi'_A(\alpha)C(G^-(\alpha)) + \pi'_B(\alpha)C(1 - G(\alpha)) \tag{37}$$

(for derivation see equation 8 in the text). The condition that the investor cannot reduce his liquidation costs by lowering α is then ($\lim_{z\to\alpha_0^-} EL'(z) \leq 0$)

$$\pi'_A(\alpha_0)C(G^-(\alpha_0)) + \pi'_B(\alpha_0)C(1 - G^-(\alpha_0)) \leq 0. \tag{38}$$

And the condition that an investor cannot reduce the costs by increasing α is ($\lim_{z\to\alpha_0^+} EL'(z) \geq 0$)

$$\pi'_A(\alpha_0)C(G(\alpha_0)) + \pi'_B(\alpha_0)C(1 - G(\alpha_0)) \geq 0. \tag{39}$$

Since $\pi'_A < 0$, $\pi'_B > 0$ and $G^-(\alpha_0) < G(\alpha_0)$, we have $\pi'_A(\alpha_0)C(G^-(\alpha_0)) > \pi'_A(\alpha_0)C(G(\alpha_0))$ and $\pi'_B(\alpha_0)C(1-G^-(\alpha_0)) > \pi'_B(\alpha_0)C(1-G(\alpha_0))$, and hence (38) and (39) form a contradiction. ■

Proof of Proposition 3. (i): To show uniqueness recall that $\pi'_A < 0$ and $\pi'_B > 0$. Since C is (strictly) increasing, the left hand side of (10) is (strictly) decreasing in G^*. Thus, (10) gives us for any $\alpha \in [0,1)$ a unique $G^*(\alpha)$. Uniqueness at $\alpha = 1$ follows from the requirement that $G(1) = 1$.

(ii): In order to show that $G^*(\alpha)$ is indeed strictly increasing, we totally differentiate the equilibrium condition (10) with respect to α

$$\pi'_A C'(G^*)G^{*\prime} + \pi''_A C(G^*) - \pi'_B C'(1-G^*)G^{*\prime} + \pi''_B C(1-G^*) = 0. \tag{40}$$

Solving for $G^{*\prime}$ gives

$$G^{*\prime} = \frac{\pi''_A C(G^*) + \pi''_B C(1-G^*)}{\pi'_B C'(1-G^*) - \pi'_A C'(G^*)}. \tag{41}$$

The nominator of (41) is (strictly) positive since $C(G^*(\alpha)) > 0$ or $C(1 - G^*(\alpha)) > 0$ (because we must have for all α that $G^* > 0$ or $1 - G^* > 0$) and since $\pi''_A, \pi''_B > 0$. The denominator is also positive since $C' > 0$, $\pi_A' < 0$ and $\pi_B' > 0$. It follows that G^* is strictly increasing.

(iii): We show that $G^*(0) > 0$ (the point mass at $\alpha = 1$ follows then from the symmetry of the problem). Suppose that we have $G^*(0) \leq 0$. Since G^* is increasing on $[0, 1)$ and does so smoothly, there exists an α_0 for which $G(\alpha_0) = 0$. For this α_0 the equilibrium condition (10) is not fulfilled since $C(G(\alpha_0)) = 0$, $C(1 - G(\alpha_0)) = C(1) > 0$ and $\pi_B' > 0$.

(iv): We show that there cannot be an equilibrium where G is not strictly increasing (that is, not strictly increasing on at least some interval). First, recall that we have already shown that G^* must be smooth on $(0, 1)$ (Proposition 2). We first show that there is density at $\alpha = 0$ and $\alpha = 1$ (that is some investors play these points). Without loss of generalization focus on $\alpha = 0$. Suppose to the contrary that there is no density at $\alpha = 0$. Denote the smallest α at which there is positive density with $\alpha_1 > 0$. We hence have that $G(\alpha) = 0$ for $\alpha < \alpha_1$ but that at least one investor plays α_1. For this investor it should not be profitable to deviate to $\alpha = 0$. The difference in expected losses between playing α_1 and 0 is given by

$$\begin{aligned} EL(\alpha_1) - EL(0) &= \int_0^d \int_x^{\widehat{y}_{\alpha_1}(x)} \phi(x)\phi(y)C(G(\widehat{\alpha}(x, y)))dydx \\ &+ \int_0^d \int_0^x \phi(x)\phi(y)C(1 - G^-(\widehat{\alpha}(x, y)))dydx \\ &+ \int_d^{\widehat{x}_{\alpha_1}(0)} \int_0^{\widehat{y}_{\alpha_1}(x)} \phi(x)\phi(y)C(1 - G^-(\widehat{\alpha}(x, y)))dydx \\ &- \int_0^d \int_x^{\infty} \phi(x)\phi(y)C(G(\widehat{\alpha}(x, y)))dydx \\ &- \int_0^d \int_0^x \phi(x)\phi(y)C(1 - G^-(\widehat{\alpha}(x, y)))dydx. \end{aligned} \tag{42}$$

This simplifies to

$$\begin{aligned} &\int_d^{\widehat{x}_{\alpha_1}(0)} \left(\int_0^{\widehat{y}_{\alpha_1}(x)} (\phi(x)\phi(y)C(1 - G(\widehat{\alpha}(x, y)))) \, dy \right) dx \\ &- \int_0^d \left(\int_{\widehat{y}_{\alpha_1}(x)}^{\infty} (\phi(x)\phi(y)C(G(\widehat{\alpha}(x, y)))) \, dy \right) dx. \end{aligned} \tag{43}$$

Note that $\widehat{\alpha}(x, y)$ in the integrals only varies between 0 and α_1. Hence, we have $G(\widehat{\alpha}(x, y)) = 0$ in the integrals. It follows that (43) can be simplified to

$$EL(\alpha_1) - EL(0) = C(1) \int_d^{\widehat{x}_{\alpha_1}(0)} \int_0^{\widehat{y}_{\alpha_1}(x)} \phi(x)\phi(y)dydx > 0. \tag{44}$$

Hence it would be profitable for the investor to deviate to $\alpha = 0$ and we have a contradiction.

Thus, we know that $\alpha = 0$ and $\alpha = 1$ are both played. We show next that there cannot be an interval without mass on $(0,1)$. From this it follows that there is positive mass everywhere (and G strictly increasing). Suppose, to the contrary, that there is such an interval. We can then extend this interval until we reach the first α (on either side) played by some investors. This is possible since we have shown that there is density at $\alpha = 0$ and $\alpha = 1$. Denote the $\alpha's$ on the lower and upper end of this interval with $\underline{\alpha}$ and $\overline{\alpha}$ ($\underline{\alpha} < \overline{\alpha}$). We thus have that $G'(\alpha) = 0$ for $\alpha \in (\underline{\alpha}, \overline{\alpha})$ and $G'(\underline{\alpha}), G'(\overline{\alpha}) > 0$ (infinite if there is a point mass).

At $\underline{\alpha}$ the expected liquidation costs should be non-decreasing in α, since otherwise an investor that plays $\underline{\alpha}$ could strictly improve his payoff by slightly increasing α. Using equation (10), this condition writes

$$\pi'_A(\underline{\alpha})C(G(\underline{\alpha})) + \pi'_B(\underline{\alpha})C(1 - G(\underline{\alpha})) \geq 0. \tag{45}$$

Likewise, an investor that plays $\overline{\alpha}$ should not be able to reduce his liquidation costs by slightly reducing α. This condition writes

$$\pi'_A(\overline{\alpha})C(G(\overline{\alpha})) + \pi'_B(\overline{\alpha})C(1 - G(\overline{\alpha})) \leq 0. \tag{46}$$

Since $\pi''_A, \pi''_B > 0$ we have $\pi'_A(\underline{\alpha}) < \pi'_A(\overline{\alpha})$ and $\pi'_B(\underline{\alpha}) < \pi'_B(\overline{\alpha})$. Noting that $G(\underline{\alpha}) = G(\overline{\alpha})$ (since there is no density on $(\underline{\alpha}, \overline{\alpha})$), we then have $\pi'_A(\underline{\alpha})C(G(\underline{\alpha})) < \pi'_A(\overline{\alpha})C(G(\overline{\alpha}))$ and $\pi'_B(\underline{\alpha})C(1 - G(\underline{\alpha})) < \pi'_B(\overline{\alpha})C(1 - G(\overline{\alpha}))$. Thus, equations (45) and (46) form a contradiction. ■

Figure 1: The Liquidation Areas

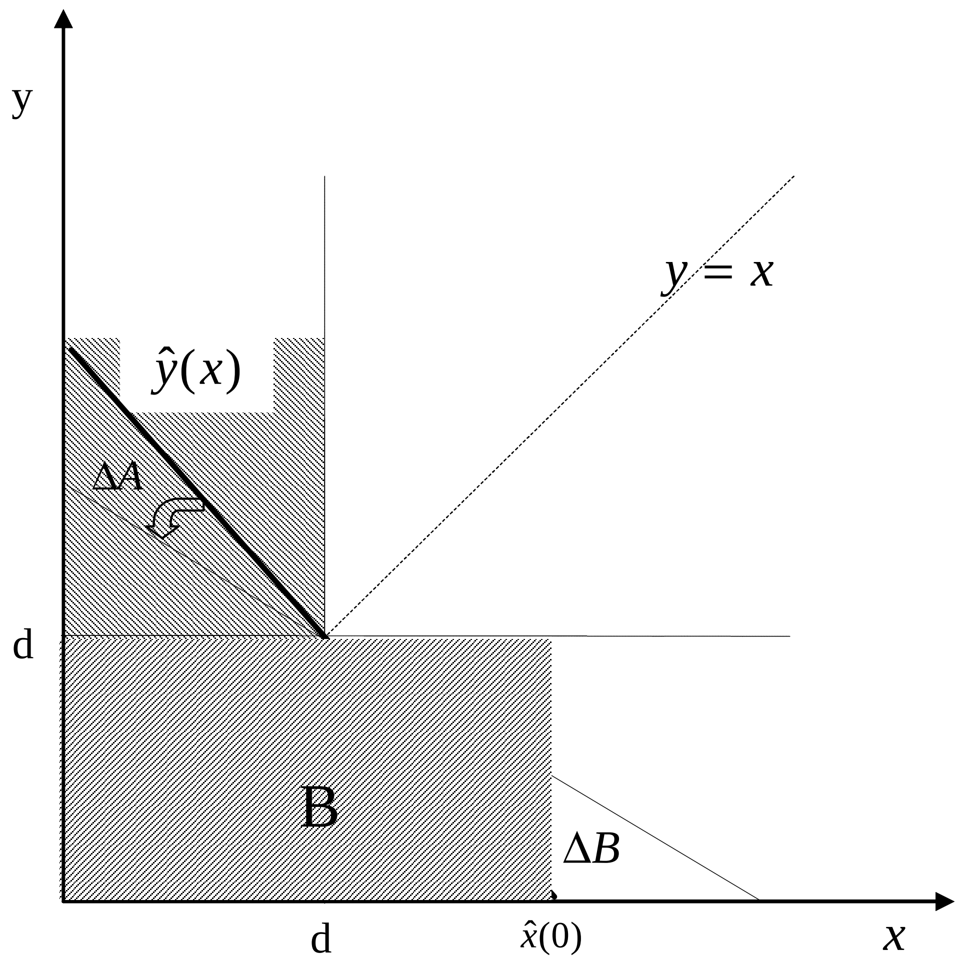

Figure 2: Equilibrium Portfolio Allocation