Multiperiod Market Equilibrium

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Introduction

- The first order conditions from an individual’s multiperiod consumption and portfolio choice problem can be interpreted as equilibrium conditions for asset pricing.

- A particular equilibrium asset pricing model where asset supplies are exogenous is the Lucas (1978) endowment economy.

- We also consider bubbles: nonfundamental asset price dynamics.
Asset Pricing in the Multiperiod Model

- In the multiperiod model, the individual’s objective is

\[ \max_{C_s, \{\omega_{is}\}, \forall s, i} E_t \left[ \sum_{s=t}^{T-1} U(C_s, s) + B(W_T, T) \right] \]  

which is solved as a series of single period problems using the Bellman equation:

\[ J(W_t, t) = \max_{C_t, \{\omega_{i,t}\}} U(C_t, t) + E_t [J(W_{t+1}, t + 1)] \]  

- This led to the first-order conditions

\[ U_C(C^*_t, t) = R_{f,t} E_t [J_{W}(W_{t+1}, t + 1)] \]

\[ = J_{W}(W_t, t) \]  

\[ E_t [R_{it} J_{W}(W_{t+1}, t + 1)] = R_{f,t} E_t [J_{W}(W_{t+1}, t + 1)], \quad i = 1, \ldots, n \]
Multiperiod Pricing Kernel

- This model has equilibrium implications even if assumptions about utility, income, and asset return distributions do not lead to explicit formulas for $C_t^*$ and $\omega_{it}^*$.

- Substituting the envelope condition $U_C (C_t^*, t) = J_W (W_t, t)$ at $t + 1$ into the right-hand side of the first line of (3),

  $$U_C (C_t^*, t) = R_{f,t} E_t [J_W (W_{t+1}, t + 1)]$$

  $$= R_{f,t} E_t [U_C (C_{t+1}^*, t + 1)]$$

  (5)

- Furthermore, substituting (4) into (3) and, again, using the envelope condition at date $t + 1$ allows us to write

  $$U_C (C_t^*, t) = E_t [R_{it} J_W (W_{t+1}, t + 1)]$$

  $$= E_t [R_{it} U_C (C_{t+1}^*, t + 1)]$$

  (6)

or
1 = E_t [m_{t,t+1} R_{it}]
= R_{f,t} E_t [m_{t,t+1}] \tag{7}

where \( m_{t,t+1} \equiv U_C (C_{t+1}^*, t + 1) / U_C (C_t^*, t) \) is the SDF (pricing kernel) between dates \( t \) and \( t + 1 \).

The relationship derived in the single-period context holds more generally: Updating equation (6) for risky asset \( j \) one period, \( U_C (C_{t+1}^*, t + 1) = E_{t+1} [R_{j,t+1} U_C (C_{t+2}^*, t + 2)] \), and substituting in the right-hand side of the original (6), one obtains

\[
U_C (C_t^*, t) = E_t \left[ R_{it} E_{t+1} \left[ R_{j,t+1} U_C (C_{t+2}^*, t + 2) \right] \right]
= E_t \left[ R_{it} R_{j,t+1} U_C (C_{t+2}^*, t + 2) \right] \tag{8}
\]
or

\[ 1 = E_t \left[ R_{it} R_{j,t+1} m_{t,t+2} \right] \]  \hspace{1cm} (9)

where \( m_{t,t+2} \equiv U_C \left( C^*_{t+2}, t + 2 \right) / U_C \left( C^*_t, t \right) \) is the marginal rate of substitution, or the SDF, between dates \( t \) and \( t + 2 \).

- By repeated substitution, (9) can be generalized to

\[ 1 = E_t \left[ R_{t,t+k} m_{t,t+k} \right] \]  \hspace{1cm} (10)

where \( m_{t,t+k} \equiv U_C \left( C^*_{t+k}, t + k \right) / U_C \left( C^*_t, t \right) \) and \( R_{t,t+k} \) is the return from any trading strategy involving multiple assets over the period from dates \( t \) to \( t + k \).

- Equation (10) says that equilibrium expected marginal utilities are equal across all time periods and assets.

- These moment conditions are often tested using a Generalized Method of Moments technique.
Lucas Model of Asset Pricing

- Lucas (1978) derives equilibrium asset prices for an endowment economy where the random process generating the economy’s real output is exogenous.
- Output obtained at a particular date cannot be reinvested and must be consumed immediately.
- Assets representing ownership claims on this exogenous output are in fixed supply.
- Assuming all individuals are identical (representative), the endowment economy assumptions fix the process for individual consumption, making the SDF exogenous.
- Exogenous output implies the market portfolio’s payout (dividend) is exogenous, which makes it easy to solve for the equilibrium price.
Including Dividends in Asset Returns

- Let the return on the $i^{th}$ risky asset, $R_{it}$, include a dividend payment made at date $t + 1$, $d_{i,t+1}$, along with a capital gain, $P_{i,t+1} - P_{it}$:

$$R_{it} = \frac{d_{i,t+1} + P_{i,t+1}}{P_{it}}$$  \hspace{1cm} (11)

- Substituting (11) into (7) and rearranging gives

$$P_{it} = E_t \left[ \frac{U_C (C^*_t, t + 1)}{U_C (C^*_t, t)} (d_{i,t+1} + P_{i,t+1}) \right]$$  \hspace{1cm} (12)

- Similar to what was done in equation (8), substitute for $P_{i,t+1}$ using equation (12) updated one period to solve forward this equation.
Including Dividends in Asset Returns cont’d

\[ P_{it} = E_t \left[ \frac{UC(C_{t+1}, t+1)}{UC(C_t^*, t)} \left( d_{i,t+1} \left( \frac{UC(C_{t+2}, t+2)}{UC(C_{t+1}, t+1)} \right) \right) \right] \]

\[ = E_t \left[ \frac{UC(C_{t+1}, t+1)}{UC(C_t^*, t)} d_{i,t+1} + \frac{UC(C_{t+2}, t+2)}{UC(C_t^*, t)} \right] (d_{i,t+2} + P_{i,t+2}) \]  

(13)

Repeating this type of substitution, that is, solving forward the difference equation (13), gives us

\[ P_{it} = E_t \left[ \sum_{j=1}^{T} \frac{UC(C_{t+j}, t+j)}{UC(C_t^*, t)} d_{i,t+j} + \frac{UC(C_{t+T}, t + T)}{UC(C_t^*, t)} P_{i,t+T} \right] \]

(14)

where the integer \( T \) reflects a large number of future periods.
Including Time Preference $\delta$

- If utility is of the form $U(C_t, t) = \delta^t u(C_t)$, where $\delta = \frac{1}{1+\rho} < 1$, then (14) becomes

$$P_{it} = E_t \left[ \sum_{j=1}^{T} \delta^j \frac{u_C(C_{t+j}^*)}{u_C(C_t^*)} d_{i,t+j} + \delta^T \frac{u_C(C_{t+T}^*)}{u_C(C_t^*)} P_{i,t+T} \right]$$

(15)

- Assuming individuals that have infinite lives or a bequest motive and $\lim_{T \to \infty} E_t \left[ \delta^T \frac{u_C(C_{t+T}^*)}{u_C(C_t^*)} P_{i,t+T} \right] = 0$ (no speculative price “bubbles”), then

$$P_{it} = E_t \left[ \sum_{j=1}^{\infty} \delta^j \frac{u_C(C_{t+j}^*)}{u_C(C_t^*)} d_{i,t+j} \right]$$

(16)
Dividends and Consumption

- In terms of the SDF \( m_{t, t+j} \equiv \delta^j u_C \left( C_{t+j}^* \right) / u_C (C_t^*) \):

\[
P_{it} = \mathbb{E}_t \left[ \sum_{j=1}^{\infty} m_{t, t+j} d_{i, t+j} \right] \tag{17}
\]

- Lucas (1978) makes (17) into a general equilibrium model by assuming an infinitely-lived *representative* individual and where risky asset \( i \) pays a real dividend of \( d_{it} \) at date \( t \).

- The dividend is nonstorable and non-reinvestable. With no wage income, aggregate consumption equals the total dividends paid by all of the \( n \) assets at that date:

\[
C_t^* = \sum_{i=1}^{n} d_{it} \tag{18}
\]
Examples

- If the representative individual is risk-neutral, so that \( u(C) = C \) and \( u_C \) is a constant (1), then (17) becomes

\[
P_{it} = E_t \left[ \sum_{j=1}^{\infty} \delta^j d_{i,t+j} \right]
\]  

(19)

- If utility is logarithmic \( u(C_t) = \ln C_t \) and aggregate dividend \( d_t = \sum_{i=1}^{n} d_{it} \), the price of risky asset \( i \) is given by

\[
P_{it} = E_t \left[ \sum_{j=1}^{\infty} \delta^j \frac{C_t^*}{C_{t+j}} d_{i,t+j} \right]
\]

\[
= E_t \left[ \sum_{j=1}^{\infty} \delta^j \frac{d_t}{d_{t+j}} d_{i,t+j} \right]
\]  

(20)
Under logarithmic utility, we can price the market portfolio without making assumptions regarding the distribution of $d_{it}$ in (20).

If $P_t$ is the value of aggregate dividends, then from (20):

$$P_t = E_t \left[ \sum_{j=1}^{\infty} \delta^j \frac{d_t}{d_{t+j}} d_{t+j} \right]$$

$$= d_t \frac{\delta}{1 - \delta}$$

(21)

Note that higher expected future dividends, $d_{t+j}$, are exactly offset by a lower expected marginal utility of consumption, $m_{t, t+j} = \delta^j d_t / d_{t+j}$, leaving the value of a claim on this output process unchanged.
Examples cont’d

- For more general power utility, \( u(C_t) = C_t^\gamma / \gamma \), we have

\[
P_t = E_t \left[ \sum_{j=1}^{\infty} \delta^j \left( \frac{d_{t+j}}{d_t} \right)^{\gamma-1} d_{t+j} \right] \\
= d_t^{1-\gamma} E_t \left[ \sum_{j=1}^{\infty} \delta^j d_{t+j}^\gamma \right] \\
= P_{ft} = \frac{1}{R_{ft}} = \delta E_t \left[ \left( \frac{d_{t+1}}{d_t} \right)^{\gamma-1} \right]
\]

which does depend on the distribution of future dividends.

- The value of a hypothetical riskless asset that pays a one-period dividend of $1 is

\[
P_{ft} = \frac{1}{R_{ft}} = \delta E_t \left[ \left( \frac{d_{t+1}}{d_t} \right)^{\gamma-1} \right]
\]
Examples cont’d

- We can view the Mehra and Prescott (1985) finding in its true multiperiod context: they used equations such as (22) and (23) with $d_t = C_t^*$ to see if a reasonable value of $\gamma$ produces a risk premium (excess average return over a risk-free return) that matches that of market portfolio of U.S. stocks’ historical average excess returns.

- Reasonable values of $\gamma$ could not match the historical risk premium of 6%, a result they described as the *equity premium puzzle*.

- As mentioned earlier, for reasonable levels of risk aversion, aggregate consumption appears to vary too little to justify the high Sharpe ratio for the market portfolio of stocks.

- The moment conditions in (22) and (23) require a highly negative value of $\gamma$ to fit the data.
Labor Income

- We can add labor income to the market endowment (Cecchetti, Lam and Mark, 1993). Human capital pays a wage payment of $y_t$ at date $t$, also non-storable. Hence, equilibrium aggregate consumption equals

\[ C^*_t = d_t + y_t \]  \hspace{1cm} (24)

so that equilibrium consumption no longer equals dividends.

- The value of the market portfolio is:

\[
P_t = E_t \left[ \sum_{j=1}^{\infty} \delta^j \frac{u_C \left( C^*_{t+j} \right)}{u_C \left( C^*_t \right)} d_{t+j} \right]
\]

\[ = E_t \left[ \sum_{j=1}^{\infty} \delta^j \left( \frac{C^*_{t+j}}{C^*_t} \right)^{\gamma^{-1}} d_{t+j} \right] \]  \hspace{1cm} (25)
Now specify separate lognormal processes for dividends and consumption:

\[
\ln \left( \frac{C_{t+1}^*}{C_t^*} \right) = \mu_c + \sigma_c \eta_{t+1} \\
\ln \left( \frac{d_{t+1}}{d_t} \right) = \mu_d + \sigma_d \varepsilon_{t+1}
\]

(26)

where the error terms are serially uncorrelated and distributed as

\[
\begin{pmatrix}
\eta_t \\
\varepsilon_t
\end{pmatrix}
\sim N
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\begin{pmatrix}
1 & \rho \\
\rho & 1
\end{pmatrix}
\]

(27)

Now what is the equilibrium price of the market portfolio?
When $\delta e^\alpha < 1$, the expectation in (25) equals

$$P_t = d_t \frac{\delta e^\alpha}{1 - \delta e^\alpha}$$  \hspace{1cm} (28)$$

where

$$\alpha \equiv \mu_d - (1 - \gamma) \mu_c + \frac{1}{2} \left[ (1 - \gamma)^2 \sigma_c^2 + \sigma_d^2 \right] - (1 - \gamma) \rho \sigma_c \sigma_d$$  \hspace{1cm} (29)$$

(See Exercise 6.3.)

Equation (28) equals (21) when $\gamma = 0$, $\mu_d = \mu_c$, $\sigma_c = \sigma_d$, and $\rho = 1$, which is the special case of log utility and no labor income.

With no labor income ($\mu_d = \mu_c$, $\sigma_c = \sigma_d$, $\rho = 1$) but $\gamma \neq 0$, we have $\alpha = \gamma \mu_c + \frac{1}{2} \gamma^2 \sigma_c^2$, which is increasing in the growth rate of dividends (and consumption) when $1 > \gamma > 0.$
Labor Income cont’d

- When $\gamma > 0$, greater dividend growth leads individuals to desire increased savings due to high intertemporal elasticity ($\varepsilon = 1/(1 - \gamma) > 1$). Market clearing requires the value of the market portfolio to rise, raising income or wealth to make desired consumption rise to equal the fixed supply.

- The reverse occurs when $\gamma < 0$, as the income or wealth effect will exceed the substitution effect.

- For the general case of labor income where $\alpha$ is given by equation (29), a lower correlation between consumption and dividends (decline in $\rho$) increases $\alpha$.

- Since $\partial P_t / \partial \alpha > 0$, lower correlation raises the value of the market portfolio because it is a better hedge against uncertain labor income.
Rational Asset Price Bubbles

- Define $p_t \equiv P_{it} u_C (C_t)$, the product of the asset price and the marginal utility of consumption.
- Then equation (12) is

$$E_t [p_{t+1}] = \delta^{-1} p_t - E_t \left[ u_C (C^*_{t+1}) d_{i,t+1} \right]$$

(30)

where $\delta^{-1} = 1 + \rho > 1$ where $\rho$ is the subjective rate of time preference. The solution (17) to this equation is referred to as the fundamental solution, which we denote as $f_t$:

$$p_t = f_t \equiv E_t \left[ \sum_{j=1}^{\infty} \delta^j u_C (C^*_{t+j}) d_{i,t+j} \right]$$

(31)

- The sum in (31) converges if the marginal utility-weighted dividends are expected to grow more slowly than the time preference discount factor.
There are other solutions to (30) of the form $p_t = f_t + b_t$ where the *bubble* component $b_t$ is any process that satisfies

$$E_t [b_{t+1}] = \delta^{-1} b_t = (1 + \rho) b_t$$

This is easily verified by substitution into (30):

$$E_t [f_{t+1} + b_{t+1}] = \delta^{-1} (f_t + b_t) - E_t \left[ u_C (C_{t+1}^*) d_{i,t+1} \right]$$

$$E_t [f_{t+1}] + E_t [b_{t+1}] = \delta^{-1} f_t + \delta^{-1} b_t - E_t \left[ u_C (C_{t+1}^*) d_{i,t+1} \right]$$

$$E_t [b_{t+1}] = \delta^{-1} b_t = (1 + \rho) b_t$$

where in the last line of (33) uses the fact that $f_t$ satisfies the difference equation. Since $\delta^{-1} > 1$, $b_t$ explodes in expected value:
The exploding nature of $b_t$ provides a rationale for interpreting the general solution $p_t = f_t + b_t$, $b_t \neq 0$, as a bubble solution.

Suppose that $b_t$ follows a deterministic time trend:

$$b_t = b_0 (1 + \rho)^t$$

Then the solution

$$p_t = f_t + b_0 (1 + \rho)^t$$

implies that the marginal utility-weighted asset price grows exponentially forever.
Next, consider a possibly more realistic modeling of a "bursting" bubble proposed by Blanchard (1979):

\[ b_{t+1} = \begin{cases} \frac{1+\rho}{q} b_t + e_{t+1} & \text{with probability } q \\ z_{t+1} & \text{with probability } 1 - q \end{cases} \]

with \( E_t [e_{t+1}] = E_t [z_{t+1}] = 0 \).

The bubble continues with probability \( q \) each period but "burst" with probability \( 1 - q \).

This process satisfies the condition in (32), so that \( p_t = f_t + b_t \) is again a valid bubble solution, and the expected return conditional on no crash is higher than in the infinite bubble.
Likelihood of Rational Bubbles

- Additional economic considerations may rule out many rational bubbles: consider negative bubbles where $b_t < 0$.
- From (34) individuals must expect that at some future date $\tau > t$ that $p_\tau = f_\tau + b_\tau$ can be negative.
- Since marginal utility is always positive, $P_{it} = p_t/u_C(C_t)$, must be expected to become negative, which is inconsistent with limited-liability securities.
- Similarly, bubbles that burst and start again can be ruled out.

Note that a general process for a bubble can be written as

$$b_t = (1 + \rho)^{t} b_0 + \sum_{s=1}^{t} (1 + \rho)^{t-s} \varepsilon_s$$  \hspace{1cm} (38)

where $\varepsilon_s$, $s = 1, \ldots, t$ are mean-zero innovations.
Likelihood of Rational Bubbles cont’d

To avoid negative values of $b_t$ (and negative expected future prices), realizations of $\varepsilon_t$ must satisfy

$$\varepsilon_t \geq - (1 + \rho) b_{t-1}, \forall t \geq 0$$  \hspace{1cm} (39)

This is due to

$$\varepsilon_t = b_t - (1 + \rho) b_{t-1}$$

$$b_t = \varepsilon_t + (1 + \rho) b_{t-1} > 0$$

$$\varepsilon_t \geq -(1 + \rho) b_{t-1}$$

For example, if $b_t = 0$ so that a bubble does not exist at date $t$, then from (39) and the requirement that $\varepsilon_{t+1}$ have mean zero, it must be the case that $\varepsilon_{t+1} = 0$ with probability 1.

Hence, if a bubble currently does not exist, it cannot get started.
Moreover, the bursting and then restarting bubble in (37) could only avoid a negative value of $b_{t+1}$ if $z_{t+1} = 0$ with probability 1 and $e_{t+1} = 0$ whenever $b_t = 0$. Hence, this type of bubble would need to be positive on the first trading day, and once it bursts it could never restart.

Tirole (1982) considers an economy model with a finite number of agents and shows that rational individuals will not trade assets at prices above their fundamental values.

Santos and Woodford (1997) consider rational bubbles in a wide variety of economies and find only a few examples of the overlapping generations type where they can exist.

If conditions for rational bubbles are limited yet bubbles seem to occur, some irrationality may be required.
Summary

- If an asset’s dividends are modeled explicitly, the asset’s price satisfies a discounted dividend formula.

- The Lucas endowment economy takes this a step further by equating aggregate dividends to consumption, simplifying valuation of claims on aggregate dividends.

- In an infinite horizon model, rational asset price bubbles are possible but additional aspects of the economic environment can often rule them out.