Basics of Derivative Pricing

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Introduction

- Derivative securities have cashflows that derive from another “underlying” variable, such as an asset price, interest rate, or exchange rate.

- The absence of arbitrage opportunities places restrictions on the derivative’s value relative to that of its underlying asset.

- For forward contracts, no-arbitrage considerations alone may lead to an exact pricing formula.

- For options, no-arbitrage restrictions cannot determine an exact price, but only bounds on the option’s price.

- An exact option pricing formula requires additional assumptions on the probability distribution of the underlying asset’s returns (e.g., binomial).
Forward Contracts on Assets Paying Dividends

- Let $F_{0\tau}$ be the date 0 forward price for exchanging one share of an underlying asset $\tau$ periods in the future. This price is agreed to at date 0 but paid at date $\tau > 0$ for delivery at date $\tau$ of the asset.

- Hence, the date $\tau > 0$ payoff to the long (short) party in this forward contract is $S_\tau - F_{0\tau}$, ($F_{0\tau} - S_\tau$) where $S_\tau$ is the date $\tau$ spot price of one share of the underlying asset.

- The parties set $F_{0\tau}$ to make the date 0 contract’s value equal 0 (no payment at date 0).

- Let $R_f > 1$ be the per-period risk-free return for borrowing or lending over the period from date 0 to date $\tau$, and let $D$ be the date 0 present value of dividends paid by the underlying asset over the period from date 0 to date $\tau$. 

### Forward Contract Cash Flows

Consider a long forward contract and the trades that would exactly replicate its date $\tau$ payoffs:

<table>
<thead>
<tr>
<th>Date 0 Trade</th>
<th>Date 0 Cashflow</th>
<th>Date $\tau$ Cashflow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long Forward Contract</td>
<td>0</td>
<td>$S_\tau - F_{0\tau}$</td>
</tr>
</tbody>
</table>

**Replicating Trades**

1. **Buy Asset and Sell Dividends**
   
   $-S_0 + D$
   
   $S_\tau$

2. **Borrow**
   
   $R_f^{-\tau} F_{0\tau}$
   
   $-F_{0\tau}$

**Net Cashflow**

$-S_0 + D + R_f^{-\tau} F_{0\tau}$

$S_\tau - F_{0\tau}$

In the absence of arbitrage, the cost of the replicating trades equals the zero cost of the long position:

$$S_0 - D - R_f^{-\tau} F_{0\tau} = 0 \quad (1)$$

or

$$F_{0\tau} = (S_0 - D) R_f^\tau \quad (2)$$
Forward Contract Replication

- If the contract had been initiated at a previous date, say date \(-1\), at the forward price \(F_{-1} = X\), then the date 0 value (replacement cost) of the long party’s payoff, say \(f_0\), would still be the cost of replicating the two cashflows:

\[
f_0 = S_0 - D - R_f^{-\tau} X \tag{3}
\]

- The forward price in equation (2) did not require an assumption regarding the random distribution of the underlying asset price, \(S_\tau\), because it was a static replication strategy.

- Replicating option payoffs will entail, in general, a dynamic replication strategy requiring distributional assumptions.
Basic Characteristics of Option Prices

- The owner of a call option has the right to buy an asset in the future at a pre-agreed price, called the exercise or strike price.
- Since the option owner’s payoff is always non-negative, this buyer must make an initial payment to the seller.
- A European option can be exercised only at the maturity of the option contract.
- Let $S_0$ and $S_\tau$ be the current and maturity date prices per share of the underlying asset, $X$ be the exercise price, and $c_t$ and $p_t$ be the date $t$ prices of European call and put options, respectively.
- Then the maturity values of European call and put options are

$$c_\tau = \max [S_\tau - X, 0] \quad (4)$$

$$p_\tau = \max [X - S_\tau, 0] \quad (5)$$
Lower Bounds on European Option Values

- Recall that the long (short) party’s payoff of a forward contract is $S_T - F_{0T} (F_{0T} - S_T)$.
- If $F_{0T}$ is like an option’s strike, $X$, then assuming $X = F_{0T}$ implies the payoff of a call (put) option weakly dominates that of a long (short) forward.
- Because equation (3) is the current value of a long forward position contract, the European call’s value must satisfy

$$c_0 \geq f_0 = S_0 - D - R_f^{-\tau} X$$

(6)

Furthermore, combining $c_0 \geq 0$ with (6) implies

$$c_0 \geq \max [S_0 - D - R_f^{-\tau} X, 0]$$

(7)

- By a similar argument,

$$p_0 \geq \max [-f_0, 0] = \max [R_f^{-\tau} X + D - S_0, 0]$$

(8)
Put-Call Parity

Put-call parity links options written on the same underlying, with the same maturity date, and exercise price.

\[ c_0 + R_f^{-\tau} X + D = p_0 + S_0 \] (9)

Consider forming the following two portfolios at date 0:

1. Portfolio A = a put option having value \( p_0 \) and a share of the underlying asset having value \( S_0 \)
2. Portfolio B = a call option having value \( c_0 \) and a bond with initial value of \( R_f^{-\tau} X + D \)

Then at date \( \tau \), these two portfolios are worth:

- Portfolio A =
  \[ \max [X - S_\tau, 0] + S_\tau + DR_f^\tau = \max [X, S_\tau] + DR_f^\tau \]
- Portfolio B = \( \max [0, S_\tau - X] + X + DR_f^\tau = \max [X, S_\tau] + DR_f^\tau \)
American Options

- An American option is at least as valuable as its corresponding European option because of its early exercise right.
- Hence if $C_0$ and $P_0$, the current values of American options, then $C_0 \geq c_0$ and $P_0 \geq p_0$.
- Some American options’ early exercise feature has no value.
- Consider a European call option on a non-dividend-paying asset, and recall that $c_0 \geq S_0 - R^{-\tau} X$.
- An American call option on the same asset exercised early is worth $C_0 = S_0 - X < S_0 - R^{-\tau} X < c_0$, a contradiction.
- For an American put option, selling the asset immediately and receiving $X$ now may be better than receiving $X$ at date $\tau$ (which has a present value of $R^{-\tau} X$). At exercise $P_0 = X - S_0$ may exceed $R^{-\tau} X + D - S_0$ if remaining dividends are small.
The no-arbitrage assumption alone cannot determine an exact option price as a function of the underlying asset.

However, particular distributional assumptions for the underlying asset can allow the option’s payoff to be replicated by trading in the underlying asset and a risk-free asset.

Cox, Ross, and Rubinstein (1979) developed a binomial model to value a European option on a non-dividend-paying stock.

The model assumes that the current stock price, $S$, either moves up by a proportion $u$, or down by a proportion $d$, each period. The probability of an up move is $\pi$. 
Binomial Option Pricing cont’d

\[ uS \quad \text{with probability } \pi \]

\[ dS \quad \text{with probability } 1 - \pi \]  \hspace{1cm} (10)

- Let \( R_f \) be one plus the risk-free rate for the period, where in the absence of arbitrage \( d < R_f < u \).
- Let \( c \) equal the current value of a European call option written on the stock and having a strike price of \( X \), so that its payoff at maturity \( \tau \) equals \( \max[0, S_\tau - X] \).
- Thus, \textit{one period prior to maturity}: 
Binomial Option Pricing cont’d

\[ c_u \equiv \max [0, uS - X] \text{ with probability } \pi \]

\[ c_d \equiv \max [0, dS - X] \text{ with probability } 1 - \pi \quad (11) \]

- To value \( c \), consider a portfolio containing \( \Delta \) shares of stock and \( \$B \) of bonds so that its current value is \( \Delta S + B \).
- This portfolio’s value evolves over the period as

\[ \Delta uS + R_f B \text{ with probability } \pi \]

\[ \Delta S + B \quad \Delta dS + R_f B \text{ with probability } 1 - \pi \quad (12) \]
Binomial Option Pricing cont’d

- With two securities (bond and stock) and two states (up or down), $\Delta$ and $B$ can be chosen to replicate the option’s payoffs:

  \[
  \Delta uS + R_f B = c_u \\
  \Delta dS + R_f B = c_d
  \]  
  \[ (13) \]

  \[ (14) \]

- Solving for $\Delta$ and $B$ that satisfy these two equations:

  \[
  \Delta^* = \frac{c_u - c_d}{(u - d) S} \\
  B^* = \frac{uc_d - dc_u}{(u - d) R_f}
  \]  
  \[ (15) \]

  \[ (16) \]

- Hence, a portfolio of $\Delta^*$ shares of stock and $B^*$ of bonds produces the same cashflow as the call option.
Therefore, the absence of arbitrage implies

\[ c = \Delta^* S + B^* \]  \hspace{1cm} (17)

where \( \Delta^* \) is the option’s *hedge ratio* and \( B^* \) is the debt financing that are positive/negative (*negative/positive*) for calls (*puts*).

*Example*: If \( S = $50, \ u = 2, \ d = .5, \ R_f = 1.25, \) and \( X = $50, \) then \( uS = $100, \ dS = $25, \ c_u = $50, \ c_d = $0. \)

Therefore,

\[ \Delta^* = \frac{50 - 0}{(2 - .5) 50} = \frac{2}{3} \]
Binomial Option Pricing cont’d

\[ B^* = \frac{0 - 25}{(2 - 0.5) \times 1.25} = -\frac{40}{3} \]

so that

\[ c = \Delta^* S + B^* = \frac{2}{3} (50) - \frac{40}{3} = \frac{60}{3} = \$20 \]

- This option pricing formula can be rewritten:

\[ c = \Delta^* S + B^* = \frac{c_u - c_d}{u - d} + \frac{uc_d - dc_u}{(u - d) R_f} \tag{18} \]

\[ = \frac{\left[ \frac{R_f - d}{u - d} \max [0, uS - X] + \frac{u - R_f}{u - d} \max [0, dS - X] \right]}{R_f} \]

which does not depend on the stock’s up/down probability, \( \pi \).
Since the stock’s expected rate of return equals \( u\pi + d(1 - \pi) - 1 \), it need not be known or estimated to solve for the no-arbitrage value of the option, \( c \).

However, we do need to know \( u \) and \( d \), the size of the stock’s movements per period which determine its volatility.

Note also that we can rewrite \( c \) as
\[
c = \frac{1}{R_f} \left[ \hat{\pi} c_u + (1 - \hat{\pi}) c_d \right]
\]
(19)

where \( \hat{\pi} \equiv \frac{R_f - d}{u - d} \) is the risk-neutral probability of the up state.

\( \hat{\pi} = \pi \) if individuals are risk-neutral since
\[
[u\pi + d(1 - \pi)] S = R_f S
\]
(20)

which implies that
Binomial Option Pricing cont’d

\[
\pi = \frac{R_f - d}{u - d} = \hat{\pi}
\]  \hspace{1cm} (21)

so that \(\hat{\pi}\) does equal \(\pi\) under risk neutrality.

Thus, (19) can be expressed as

\[
c_t = \frac{1}{R_f} \hat{E} [c_{t+1}]
\]  \hspace{1cm} (22)

where \(\hat{E} [\cdot]\) denotes the expectation operator evaluated using the risk-neutral probabilities \(\hat{\pi}\) rather than the true, or physical, probabilities \(\pi\).
Next, consider the option’s value with two periods prior to maturity. The stock price process is

\[ S \xrightarrow{u} uS \xrightarrow{d} duS \xrightarrow{d} d^2S \]
\[ S \xleftarrow{u} u^2S \xleftarrow{d} dS \xleftarrow{d} d^2S \]

so that the option price process is
Multiperiod Binomial Option Pricing cont’d

\[ c_{uu} \equiv \max \left[ 0, u^2 S - X \right] \]

\[ c_{du} \equiv \max \left[ 0, duS - X \right] \quad (24) \]

\[ c_{dd} \equiv \max \left[ 0, d^2 S - X \right] \]

We know how to solve one-period problems:

\[ c_u = \frac{\hat{\pi} c_{uu} + (1 - \hat{\pi}) c_{du}}{R_f} \quad (25) \]

\[ c_d = \frac{\hat{\pi} c_{du} + (1 - \hat{\pi}) c_{dd}}{R_f} \quad (26) \]
Multiperiod Binomial Option Pricing cont’d

- With two periods to maturity, the next period cashflows of $c_u$ and $c_d$ are replicated by a portfolio of $\Delta^* = \frac{c_u - c_d}{(u-d)S}$ shares of stock and $B^* = \frac{uc_d - dc_u}{(u-d)R_f}$ of bonds. No arbitrage implies

$$c = \Delta^* S + B^* = \frac{1}{R_f} [\hat{\pi} c_u + (1 - \hat{\pi}) c_d] \quad (27)$$

which, as before says that $c_t = \frac{1}{R_f} \hat{E} [c_{t+1}]$.

- The market is complete over both the last period and second-to-last periods. Substituting in for $c_u$ and $c_d$, we have

$$c = \frac{1}{R_f^2} \left[ \hat{\pi}^2 c_{uu} + 2\hat{\pi} (1 - \hat{\pi}) c_{ud} + (1 - \hat{\pi})^2 c_{dd} \right]$$
Multiperiod Binomial Option Pricing cont’d

\[ c_t = \frac{1}{R_f^2} E[c_{t+2}] \]

which says \( c_t = \frac{1}{R_f^2} E[c_{t+2}] \). Note when a market is complete each period, it becomes *dynamically complete*. By appropriate trading in just two assets, payoffs in three states of nature can be replicated.

- Repeating this analysis for any period prior to maturity, we always obtain

\[ c = \Delta^* S + B^* = \frac{1}{R_f} [\hat{\pi} c_u + (1 - \hat{\pi}) c_d] \quad (28) \]
Repeated substitution for \( c_u, c_d, c_{uu}, c_{ud}, c_{dd}, c_{uuu} \), and so on, we obtain the formula, with \( n \) periods prior to maturity:

\[
\begin{align*}
    c &= \frac{1}{R^n_f} \left[ \sum_{j=0}^{n} \left( \frac{n!}{j! (n-j)!} \right) \pi^j (1 - \pi)^{n-j} \max[0, u^j d^{n-j} S - X] \right] \\
    \text{(29)}
\end{align*}
\]

or \( c_t = \frac{1}{R^n_f} \hat{E}[c_{t+n}] \). Define “\( a \)” as the minimum number of upward jumps of \( S \) for it to exceed \( X \).

Then for all \( j < a \) (out of the money):

\[
\max[0, u^j d^{n-j} S - X] = 0 \\
\text{(30)}
\]

while for all \( j > a \) (in the money):

\[
\max[0, u^j d^{n-j} S - X] = u^j d^{n-j} S - X \\
\text{(31)}
\]
Thus, the formula for $c$ can be simplified:

$$c = \frac{1}{R_f^n} \left[ \sum_{j=a}^{n} \left( \frac{n!}{j! (n-j)!} \right) \hat{\pi}^j (1 - \hat{\pi})^{n-j} \left[ u^j d^{n-j} S - X \right] \right]$$

(32)

Breaking up (32) into two terms, we have

$$c = S \left[ \sum_{j=a}^{n} \left( \frac{n!}{j! (n-j)!} \right) \hat{\pi}^j (1 - \hat{\pi})^{n-j} \left[ \frac{u^j d^{n-j}}{R_f^n} \right] \right]$$

$$- X R_f^{-n} \left[ \sum_{j=a}^{n} \left( \frac{n!}{j! (n-j)!} \right) \hat{\pi}^j (1 - \hat{\pi})^{n-j} \right]$$

(33)

The terms in brackets are complementary binomial distribution functions, so that (33) can be written...
Multiperiod Binomial Option Pricing cont’d

\[ c = S \phi[a; n, \hat{\pi}'] - XR_f^{-n} \phi[a; n, \hat{\pi}] \]  \hspace{1cm} (34)

where \( \hat{\pi}' = \left( \frac{u}{R_f} \right) \hat{\pi} \) and \( \phi[a; n, \hat{\pi}] \) is the probability that the sum of \( n \) random variables that equal 1 with probability \( \hat{\pi} \) and 0 with probability \( 1 - \hat{\pi} \) is \( \geq a \).

For time to maturity \( \tau \) and per-unit variance \( \sigma^2 \) (depending on \( u \) and \( d \)), as the number of periods \( n \to \infty \), but the length of each period \( \frac{\tau}{n} \to 0 \), this formula converges to:

\[ c = SN(z) - XR_f^{-\tau} N \left( z - \sigma \sqrt{\tau} \right) \]  \hspace{1cm} (35)

where \( z \equiv \left[ \ln \left( \frac{S}{XR_f^{-\tau}} \right) + \frac{1}{2} \sigma^2 \tau \right] / (\sigma \sqrt{\tau}) \) and \( N(\cdot) \) is the cumulative standard normal distribution function.
Summary

- Forward contract payoffs can be replicated using a static trading strategy.

- Option contract payoffs require a dynamic trading strategy.

- A dynamically complete market allows us to use risk-neutral valuation.

- Dynamically complete markets imply replication of payoffs in all future states, but we may need to execute many trades to do so.