Basics of Derivative Pricing

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Introduction

- Derivative securities have cashflows that derive from another “underlying” variable, such as an asset price, interest rate, or exchange rate.
- The absence of arbitrage opportunities places restrictions on the derivative’s value relative to that of its underlying asset.
- For forward contracts, no-arbitrage considerations alone may lead to an exact pricing formula.
- For options, no-arbitrage restrictions cannot determine an exact price, but only bounds on the option’s price.
- An exact option pricing formula requires additional assumptions on the probability distribution of the underlying asset’s returns (e.g., binomial).
Forward Contracts on Assets Paying Dividends

- Let $F_{0\tau}$ be the date 0 forward price for exchanging one share of an underlying asset $\tau$ periods in the future. This price is agreed to at date 0 but paid at date $\tau > 0$ for delivery at date $\tau$ of the asset.

- Hence, the date $\tau > 0$ payoff to the long (short) party in this forward contract is $S_\tau - F_{0\tau}$, $(F_{0\tau} - S_\tau)$ where $S_\tau$ is the date $\tau$ spot price of one share of the underlying asset.

- The parties set $F_{0\tau}$ to make the date 0 contract’s value equal 0 (no payment at date 0).

- Let $R_f > 1$ be the per-period risk-free return for borrowing or lending over the period from date 0 to date $\tau$, and let $D$ be the date 0 present value of dividends paid by the underlying asset over the period from date 0 to date $\tau$. 

Consider a long forward contract and the trades that would exactly replicate its date $\tau$ payoffs:

<table>
<thead>
<tr>
<th>Date 0 Trade</th>
<th>Date 0 Cashflow</th>
<th>Date $\tau$ Cashflow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long Forward Contract</td>
<td>0</td>
<td>$S_\tau - F_{0\tau}$</td>
</tr>
</tbody>
</table>

**Replicating Trades**
1) Buy Asset and Sell Dividends
   $-S_0 + D$  
   $S_\tau$
2) Borrow
   $R_f^{-\tau} F_{0\tau}$  
   $-F_{0\tau}$

Net Cashflow
   $-S_0 + D + R_f^{-\tau} F_{0\tau}$  
   $S_\tau - F_{0\tau}$

In the absence of arbitrage, the cost of the replicating trades equals the zero cost of the long position:

$$S_0 - D - R_f^{-\tau} F_{0\tau} = 0 \quad (1)$$

or

$$F_{0\tau} = (S_0 - D) R_f^{\tau} \quad (2)$$
Forward Contract Replication

- If the contract had been initiated at a previous date, say date $-1$, at the forward price $F_{-1} = X$, then the date 0 value (replacement cost) of the long party’s payoff, say $f_0$, would still be the cost of replicating the two cashflows:

$$f_0 = S_0 - D - R_f^{-\tau} X$$  \hspace{1cm} (3)

- The forward price in equation (2) did not require an assumption regarding the random distribution of the underlying asset price, $S_\tau$, because it was a static replication strategy.

- Replicating option payoffs will entail, in general, a dynamic replication strategy requiring distributional assumptions.
Basic Characteristics of Option Prices

- The owner of a call option has the right to buy an asset in the future at a pre-agreed price, called the exercise or strike price.
- Since the option owner’s payoff is always non-negative, this buyer must make an initial payment to the seller.
- A European option can be exercised only at the maturity of the option contract.
- Let $S_0$ and $S_\tau$ be the current and maturity date prices per share of the underlying asset, $X$ be the exercise price, and $c_t$ and $p_t$ be the date $t$ prices of European call and put options, respectively.
- Then the maturity values of European call and put options are
  \[ c_\tau = \max[S_\tau - X, 0] \]  
  \[ p_\tau = \max[X - S_\tau, 0] \]
Recall that the long (short) party's payoff of a forward contract is $S_\tau - F_{0\tau} \ (F_{0\tau} - S_\tau)$.

If $F_{0\tau}$ is like an option's strike, $X$, then assuming $X = F_{0\tau}$ implies the payoff of a call (put) option weakly dominates that of a long (short) forward.

Because equation (3) is the current value of a long forward position contract, the European call's value must satisfy

$$c_0 \geq S_0 - D - R_f^{-\tau} X$$  \hspace{1cm} (6)

Furthermore, combining $c_0 \geq 0$ with (6) implies

$$c_0 \geq \max \left[ S_0 - D - R_f^{-\tau} X, 0 \right]$$  \hspace{1cm} (7)

By a similar argument,

$$p_0 \geq \max \left[ R_f^{-\tau} X + D - S_0, 0 \right]$$  \hspace{1cm} (8)
Put-Call Parity

- **Put-call parity** links options written on the same underlying, with the same maturity date, and exercise price.

\[
c_0 + R_f^{-\tau} X + D = p_0 + S_0
\]  

(9)

- Consider forming the following two portfolios at date 0:
  1. Portfolio A = a put option having value \( p_0 \) and a share of the underlying asset having value \( S_0 \)
  2. Portfolio B = a call option having value \( c_0 \) and a bond with initial value of \( R_f^{-\tau} X + D \)

Then at date \( \tau \), these two portfolios are worth:

- Portfolio A =
  \[
  \max \left[ X - S_\tau, 0 \right] + S_\tau + DR_f^\tau = \max \left[ X, S_\tau \right] + DR_f^\tau
  \]

- Portfolio B = \( \max \left[ 0, S_\tau - X \right] + \]
  \[
  X + DR_f^\tau = \max \left[ X, S_\tau \right] + DR_f^\tau
  \]
American Options

- An American option is at least as valuable as its corresponding European option because of its early exercise right.
- Hence if \( C_0 \) and \( P_0 \), the current values of American options, then \( C_0 \geq c_0 \) and \( P_0 \geq p_0 \).
- Some American options’ early exercise feature has no value.
- Consider a European call option on a non-dividend-paying asset with \( c_0 \geq S_0 - R^{-\tau}X \).
- An American call option on the same asset exercised early is worth \( C_0 = S_0 - X < S_0 - R^{-\tau}X < c_0 \), a contradiction.
- For an American put option, selling the asset immediately and receiving \( $X \) now may be better than receiving \( $X \) at date \( \tau \) (which has a present value of \( R^{-\tau}X \)). At exercise \( P_0 = X - S_0 \) may exceed \( R^{-\tau}X + D - S_0 \) if remaining dividends are small.
Binomial Option Pricing

- The no-arbitrage assumption alone cannot determine an exact option price as a function of the underlying asset.

- However, particular distributional assumptions for the underlying asset can allow the option’s payoff to be replicated by trading in the underlying asset and a risk-free asset.

- Cox, Ross, and Rubinstein (1979) binomial model can be used to value a European option on a non-dividend stock.

- It assumes that the current stock price, $S$, either moves up, by a proportion $u$, or down, by a proportion $d$, each period. The probability of an up move is $\pi$. 
Binomial Option Pricing cont’d

\[ \begin{align*}
S & \quad \text{with probability } \pi \\
S & \quad \text{with probability } 1 - \pi
\end{align*} \]  
(10)

- Let \( R_f \) be one plus the risk-free rate for the period, where in the absence of arbitrage \( d < R_f < u \).
- Let \( c \) equal the value of a European call option written on the stock and having a strike price of \( X \), so that at maturity, \( c = \max[0, S_T - X] \).
- Thus, *one period prior to maturity*:
Binomial Option Pricing cont’d

\[ c_u \equiv \max [0, uS - X] \quad \text{with probability } \pi \]

\[ c_d \equiv \max [0, dS - X] \quad \text{with probability } 1 - \pi \]  \hspace{1cm} (11)

- To value \( c \), consider a portfolio containing \( \Delta \) shares of stock and \$B\) of bonds so that its current value is \( \Delta S + B \).
- This portfolio’s value evolves over the period as

\[ \Delta uS + R_f B \quad \text{with probability } \pi \]

\[ \Delta S + B \]

\[ \Delta dS + R_f B \quad \text{with probability } 1 - \pi \]  \hspace{1cm} (12)
Binomial Option Pricing cont’d

- With two securities (bond and stock) and two states (up or down), $\Delta$ and $B$ can be chosen to replicate the option’s payoffs:

$$\Delta uS + R_f B = c_u \quad (13)$$
$$\Delta dS + R_f B = c_d \quad (14)$$

- Solving for $\Delta$ and $B$ that satisfy these two equations:

$$\Delta^* = \frac{c_u - c_d}{(u - d) S} \quad (15)$$
$$B^* = \frac{uc_d - dc_u}{(u - d) R_f} \quad (16)$$

- Hence, a portfolio of $\Delta^*$ shares of stock and $B^*$ of bonds produces the same cashflow as the call option.
Binomial Option Pricing Example

• Therefore, the absence of arbitrage implies

\[ c = \Delta^* S + B^* \]  \hspace{1cm} (17)

where \( \Delta^* \) is the option’s *hedge ratio* and \( B^* \) is the debt financing that are positive/negative (*negative/positive*) for calls (*puts*).

• *Example*: If \( S = $50, u = 2, d = .5, R_f = 1.25, \) and \( X = $50 \), then \( uS = $100, dS = $25, c_u = $50, c_d = $0. \)

• Therefore,

\[ \Delta^* = \frac{50 - 0}{(2 - .5) 50} = \frac{2}{3} \]
Binomial Option Pricing cont’d

\[ B^* = \frac{0 - 25}{(2 - .5) 1.25} = -\frac{40}{3} \]

so that

\[ c = \Delta^* S + B^* = \frac{2}{3} (50) - \frac{40}{3} = \frac{60}{3} = 20 \]

- This option pricing formula can be rewritten:

\[
c = \Delta^* S + B^* = \frac{c_u - c_d}{u - d} + \frac{uc_d - dc_u}{(u - d) R_f} \tag{18}
\]

\[
= \left[ \frac{R_f - d}{u - d} \max [0, uS - X] + \frac{u - R_f}{u - d} \max [0, dS - X] \right] R_f
\]

which *does not* depend on the stock’s up/down probability, \( \pi \).
Binomial Option Pricing cont’d

- Since the stock’s expected rate of return equals $u\pi + d(1 - \pi) - 1$, it need not be known or estimated to solve for the no-arbitrage value of the option, $c$.
- However, we do need to know $u$ and $d$, the size of the stock’s movements per period which determine its volatility.
- Note also that we can rewrite $c$ as

$$c = \frac{1}{R_f} [\hat{\pi} c_u + (1 - \hat{\pi}) c_d]$$

(19)

where $\hat{\pi} \equiv \frac{R_f - d}{u - d}$ is the risk-neutral probability of the up state.

- $\hat{\pi} = \pi$ if individuals are risk-neutral since

$$[u\pi + d (1 - \pi)] S = R_f S$$

(20)

which implies that
Binomial Option Pricing cont’d

\[ \pi = \frac{R_f - d}{u - d} = \hat{\pi} \]  \hspace{1cm} (21)

so that \( \hat{\pi} \) does equal \( \pi \) under risk neutrality.

Thus, (19) can be expressed as

\[ c_t = \frac{1}{R_f} \hat{E} [c_{t+1}] \]  \hspace{1cm} (22)

where \( \hat{E} [\cdot] \) denotes the expectation operator evaluated using the risk-neutral probabilities \( \hat{\pi} \) rather than the true, or physical, probabilities \( \pi \).
Next, consider the option’s value with *two periods prior to maturity*. The stock price process is

\[
\begin{align*}
    &u^2 S \\
    &uS \\
    &S \\
    &dS \\
    &duS \\
    &d^2 S \\
\end{align*}
\]

so that the option price process is
Multiperiod Binomial Option Pricing cont’d

\[ c_{uu} \equiv \max[0, u^2 S - X] \]

\[ c_{du} \equiv \max[0, duS - X] \quad (24) \]

\[ c_{dd} \equiv \max[0, d^2 S - X] \]

- We know how to solve one-period problems:

\[ c_u = \frac{\hat{\pi} c_{uu} + (1 - \hat{\pi}) c_{du}}{R_f} \quad (25) \]

\[ c_d = \frac{\hat{\pi} c_{du} + (1 - \hat{\pi}) c_{dd}}{R_f} \quad (26) \]
With two periods to maturity, the next period cashflows of $c_u$ and $c_d$ are replicated by a portfolio of $\Delta^* = \frac{c_u - c_d}{(u-d)S}$ shares of stock and $B^* = \frac{uc_d - dc_u}{(u-d)R_f}$ of bonds. No arbitrage implies

$$c = \Delta^* S + B^* = \frac{1}{R_f} [\hat{\pi}c_u + (1 - \hat{\pi})c_d] \quad (27)$$

which, as before says that $c_t = \frac{1}{R_f} \hat{E}[c_{t+1}]$.

The market is complete over both the last period and second-to-last periods. Substituting in for $c_u$ and $c_d$, we have

$$c = \frac{1}{R_f^2} \left[ \hat{\pi}^2 c_{uu} + 2\hat{\pi}(1 - \hat{\pi})c_{ud} + (1 - \hat{\pi})^2 c_{dd} \right]$$
Multiperiod Binomial Option Pricing cont’d

\[
\frac{1}{R_f^2} \left[ \hat{\pi}^2 \max [0, u^2 S - X] + 2\hat{\pi} (1 - \hat{\pi}) \max [0, duS - X] \right]
\]

\[
+ \frac{1}{R_f^2} \left[ (1 - \hat{\pi})^2 \max [0, d^2 S - X] \right]
\]

which says \( c_t = \frac{1}{R_f^2} \hat{E} [c_{t+2}] \). Note when a market is complete each period, it becomes \textit{dynamically complete}. By appropriate trading in just two assets, payoffs in three states of nature can be replicated.

- Repeating this analysis for any period prior to maturity, we always obtain

\[
c = \Delta^* S + B^* = \frac{1}{R_f} \left[ \hat{\pi} c_u + (1 - \hat{\pi}) c_d \right]
\]  \hspace{1cm} (28)
Repeated substitution for \( c_u, c_d, c_{uu}, c_{ud}, c_{dd}, c_{uuu} \), and so on, we obtain the formula, with \( n \) periods prior to maturity:

\[
c = \frac{1}{R_f^n} \left[ \sum_{j=0}^{n} \left( \frac{n!}{j! (n-j)!} \right) \hat{\pi}^j (1 - \hat{\pi})^{n-j} \max [0, u^j d^{n-j} S - X] \right]
\]

or \( c_t = \frac{1}{R_f^n} \hat{E} [c_{t+n}] \). Define “\( a \)” as the minimum number of upward jumps of \( S \) for it to exceed \( X \).

Then for all \( j < a \) (out of the money):

\[
\max [0, u^j d^{n-j} S - X] = 0 \quad (30)
\]

while for all \( j > a \) (in the money):

\[
\max [0, u^j d^{n-j} S - X] = u^j d^{n-j} S - X \quad (31)
\]
Thus, the formula for $c$ can be simplified:

$$c = \frac{1}{R_f^n} \left[ \sum_{j=a}^{n} \left( \frac{n!}{j! (n-j)!} \right) \hat{\pi}^j (1 - \hat{\pi})^{n-j} \left[ u^j d^{n-j} S - X \right] \right]$$

(32)

Breaking up (32) into two terms, we have

$$c = S \left[ \sum_{j=a}^{n} \left( \frac{n!}{j! (n-j)!} \right) \hat{\pi}^j (1 - \hat{\pi})^{n-j} \left[ \frac{u^j d^{n-j}}{R_f^n} \right] \right]$$

$$- X R_f^{-n} \left[ \sum_{j=a}^{n} \left( \frac{n!}{j! (n-j)!} \right) \hat{\pi}^j (1 - \hat{\pi})^{n-j} \right]$$

(33)

The terms in brackets are complementary binomial distribution functions, so that (33) can be written
Multiperiod Binomial Option Pricing cont’d

\[ c = S \phi[a; n, \hat{\pi}'] - XR_{f}^{-n} \phi[a; n, \hat{\pi}] \]  

(34)

where \( \hat{\pi}' \equiv \left( \frac{u}{R_f} \right) \hat{\pi} \) and \( \phi[a; n, \hat{\pi}] \) is the probability that the sum of \( n \) random variables that equal 1 with probability \( \hat{\pi} \) and 0 with probability \( 1 - \hat{\pi} \) is \( \geq a \).

For time to maturity \( \tau \) and per-unit variance \( \sigma^2 \) (depending on \( u \) and \( d \)), as the number of periods \( n \to \infty \), but the length of each period \( \frac{\tau}{n} \to 0 \), this formula converges to:

\[ c = SN(z) - XR_{f}^{-\tau} N(z - \sigma \sqrt{\tau}) \]  

(35)

where \( z \equiv \left[ \ln \left( \frac{S}{XR_{f}^{-\tau}} \right) + \frac{1}{2} \sigma^2 \tau \right] / (\sigma \sqrt{\tau}) \) and \( N(\cdot) \) is the cumulative standard normal distribution function.
Summary

- Forward contract payoffs can be replicated using a static trading strategy.
- Option contract payoffs require a dynamic trading strategy.
- A dynamically complete market allows us to use risk-neutral valuation.
- Dynamically complete markets imply replication of payoffs in all future states, but we may need to execute many trades to do so.