Introduction

- We cover the basic properties of continuous-time stochastic processes having continuous paths, which are used to model many financial and economic time series.
- When asset prices follow such processes, dynamically complete markets may be possible when continuous trading is permitted.
- We show how:
  - A Brownian motion is a continuous-time limit of a discrete random walk.
  - Diffusion processes can be built from Brownian motions.
  - Itô's Lemma derives the process for a function of a variable that follows a continuous-time stochastic process.
Pure Brownian Motion

- Consider the stochastic process observed at date $t$, $z(t)$.
- Let $\Delta t$ be a discrete change in time. The change in $z(t)$ over the time interval $\Delta t$ is

$$z(t + \Delta t) - z(t) \equiv \Delta z = \sqrt{\Delta t} \tilde{\epsilon}$$

where $\tilde{\epsilon}$ is a random variable with $E[\tilde{\epsilon}] = 0$, $Var[\tilde{\epsilon}] = 1$, and $Cov[z(t + \Delta t) - z(t), z(s + \Delta t) - z(s)] = 0$ if $(t, t + \Delta t)$ and $(s, s + \Delta t)$ are nonoverlapping time intervals.

- $z(t)$ is an example of a “random walk” process: $E[\Delta z] = 0$, $Var[\Delta z] = \Delta t$, and $z(t)$ has serially uncorrelated increments.

- Now consider the change in $z(t)$ over a fixed interval, from 0 to $T$. Assume $T$ is made up of $n$ intervals of length $\Delta t$. 
Pure Brownian Motion cont’d

Then

\[ z(T) - z(0) = \sum_{i=1}^{n} \Delta z_i \]  \hspace{1cm} (2)

where \( \Delta z_i \equiv z(i \cdot \Delta t) - z([i-1] \cdot \Delta t) \equiv \sqrt{\Delta t} \tilde{\epsilon}_i \), and \( \tilde{\epsilon}_i \) is the value of \( \tilde{\epsilon} \) over the \( i^{th} \) interval. Hence (2) can be written

\[ z(T) - z(0) = \sum_{i=1}^{n} \sqrt{\Delta t} \tilde{\epsilon}_i = \sqrt{\Delta t} \sum_{i=1}^{n} \tilde{\epsilon}_i \]  \hspace{1cm} (3)

Now the first two moments of \( z(T) - z(0) \) are

\[ E_0[ z(T) - z(0) ] = \sqrt{\Delta t} \sum_{i=1}^{n} E_0[ \tilde{\epsilon}_i ] = 0 \]  \hspace{1cm} (4)
Continuous-Time Limit

\[ Var_0[z(T) - z(0)] = \left( \sqrt{\Delta t} \right)^2 \sum_{i=1}^{n} Var_0[\widetilde{\epsilon}_i] = \Delta t \cdot n \cdot 1 = T \]

where \( E_t [\cdot] \) and \( Var_t [\cdot] \) are conditional on information at date \( t \).

- Given \( T \), the mean and variance of \( z(T) - z(0) \) are independent of \( n \), the number of intervals.
- Keep \( T \) fixed but let \( n \rightarrow \infty \). What do we know besides the first two moments? From the Central Limit Theorem,

\[
p \lim_{n \to \infty} (z(T) - z(0)) = p \lim_{\Delta t \to 0} (z(T) - z(0)) \sim N(0, T)
\]
Continuous-Time Limit cont’d

- Without loss of generality, assume $\tilde{\varepsilon}_i \sim N(0, 1)$. The limit of one of these minute independent increments can be defined as

\[
dz(t) \equiv \lim_{\Delta t \to 0} \Delta z = \lim_{\Delta t \to 0} \sqrt{\Delta} t\tilde{\varepsilon} \tag{6}
\]

- Hence, $E[dz(t)] = 0$ and $\text{Var}[dz(t)] = dt$, i.e., the size of the time interval as $\Delta t \to 0$: $\int_0^T dt = T$.

- $dz$ is referred to as a pure Brownian motion or Wiener process. It follows that

\[
z(T) - z(0) = \int_0^T dz(t) \sim N(0, T) \tag{7}
\]

- The integral in (7) is a stochastic or Itô integral.
Continuous-Time Limit cont’d

- $z(t)$ is a continuous process that is nowhere differentiable; $dz(t)/dt$ does not exist.
- Below is a $z(t)$ with $T = 2$ and $n = 20$, so that $\Delta t = 0.1$. As $n \to \infty$, so that $\Delta t \to 0$, $z(t)$ becomes Brownian motion.
Diffusion Processes

- Define a new process $x(t)$ by
  \[ dx(t) = \sigma \, dz(t) \] (8)

- Then over a discrete interval, $[0, T]$, $x(t)$ is distributed
  \[ x(T) - x(0) = \int_0^T \, dx = \int_0^T \sigma \, dz(t) = \sigma \int_0^T \, dz(t) \sim N(0, \sigma^2 T) \] (9)

- Next, add a deterministic (nonstochastic) change of $\mu(t)$ per unit of time to the $x(t)$ process:
  \[ dx = \mu(t) \, dt + \sigma \, dz \] (10)

- Over any discrete interval, $[0, T]$, we obtain
Diffusion Processes cont’d

\[ x(T) - x(0) = \int_0^T dx = \int_0^T \mu(t) dt + \int_0^T \sigma dz(t) \]  
\[ = \int_0^T \mu(t) dt + \sigma \int_0^T dz(t) \sim N(\int_0^T \mu(t) dt, \sigma^2 T) \]  

- If \( \mu(t) = \mu \), a constant, then
  \[ x(T) - x(0) = \mu T + \sigma \int_0^T dz(t) \sim N(\mu T, \sigma^2 T) \].
- The process \( dx = \mu dt + \sigma dz \) is arithmetic Brownian motion.
- More generally, if \( \mu \) and \( \sigma \) are functions of time, \( t \), and/or \( x(t) \), the stochastic differential equation describes \( x(t) \)
  \[ dx(t) = \mu[x(t), t] dt + \sigma[x(t), t] dz \]  
\[ \text{(12)} \]
Diffusion Processes cont’d

- It is a continuous-time Markov process with drift $\mu[x(t), t]$ and volatility $\sigma[x(t), t]$.
- Equation (12) can be rewritten as an integral equation:

$$x(T) - x(0) = \int_0^T dx = \int_0^T \mu[x(t), t] \, dt + \int_0^T \sigma[x(t), t] \, dz$$  \hspace{1cm} (13)

- $dx(t)$ is *instantaneously* normally distributed with mean $\mu[x(t), t] \, dt$ and variance $\sigma^2[x(t), t] \, dt$, but over any finite interval, $x(t)$ generally is not normally distributed.
Definition of an Itô Integral

- An Itô integral is formally defined as a mean-square limit of a sum involving the discrete $\Delta z_i$ processes. For example, the Itô integral $\int_0^T \sigma[x(t), t] \, dz$, is defined from

$$\lim_{n \to \infty} E_0 \left[ \left( \sum_{i=1}^{n} \sigma [x([i - 1] \cdot \Delta t), [i - 1] \cdot \Delta t] \Delta z_i - \int_0^T \sigma[x(t), t] \, dz \right)^2 \right] = 0 \quad (14)$$

where within the parentheses of (14) is the difference between the Itô integral and its discrete-time approximation.

- An important Itô integral is $\int_0^T [dz(t)]^2$. In this case, (14) gives its definition

$$\lim_{n \to \infty} E_0 \left[ \left( \sum_{i=1}^{n} [\Delta z_i]^2 - \int_0^T [dz(t)]^2 \right)^2 \right] = 0 \quad (15)$$
Definition of an Itô Integral cont’d

- To understand $\int_0^T [dz(t)]^2$, recall from (5) that

$$\text{Var}_0 [z(T) - z(0)] = \text{Var}_0 \left[ \sum_{i=1}^n \Delta z_i \right] = \mathbb{E}_0 \left[ \left( \sum_{i=1}^n \Delta z_i \right)^2 \right]$$

$$= \mathbb{E}_0 \left[ \sum_{i=1}^n [\Delta z_i]^2 \right] = T \quad (16)$$

because $\Delta z_i$ are serially uncorrelated.

- One can show that

$$\mathbb{E}_0 \left[ \left( \sum_{i=1}^n [\Delta z_i]^2 - T \right)^2 \right] = 2T\Delta t \quad (17)$$
Mean Square Convergence Proof

\[ E_0 \left[ \left( \sum_{i=1}^{n} [\Delta z_i]^2 - T \right)^2 \right] = \]

\[ = E_0 \left[ \sum_{i=1}^{n} [\Delta z_i]^2 \sum_{j=1}^{n} [\Delta z_j]^2 \right] - 2E_0 \left[ \sum_{i=1}^{n} [\Delta z_i]^2 \right] T + T^2 \]

\[ = E_0 \left[ \sum_{i=1}^{n} [\Delta z_i]^4 \right] + E_0 \left[ \sum_{i \neq j}^{n} [\Delta z_i]^2 [\Delta z_j]^2 \right] - 2T^2 + T^2 \]

\[ = 3n(\Delta t)^2 + (n^2 - n)(\Delta t)^2 - T^2 = 3n(\Delta t)^2 - n(\Delta t)^2 + T^2 - T^2 \]

\[ = 2(n\Delta t)\Delta t = 2T\Delta t \]

- The limit as \( \Delta t \to 0 \), or \( n \to \infty \), of (17) results in

\[ \lim_{n \to \infty} E_0 \left[ \left( \sum_{i=1}^{n} [\Delta z_i]^2 - T \right)^2 \right] = \lim_{\Delta t \to 0} 2T\Delta t = 0 \quad (18) \]
Convergence

- Comparing (15) with (18) implies that in mean-square convergence:

\[ \int_0^T [dz(t)]^2 = T \]

Since \( \int_0^T [dz(t)]^2 \) converges to \( \int_0^T dt \) for any \( T \), over an infinitesimally short time period \( [dz(t)]^2 \) converges to \( dt \).

- If \( F \) is a function of the current value of a diffusion process, \( x(t) \), and (possibly) also is a direct function of time, Itô’s lemma shows us how to characterize \( dF(x(t), t) \).
Functions of Continuous-Time Processes and Itô’s Lemma

- Itô’s lemma is the *fundamental theorem of stochastic calculus*.
- It derives the process of a function of a diffusion process.
- *Itô’s Lemma (univariate case)*: Let $x(t)$ follow the stochastic differential equation $dx(t) = \mu(x, t) \, dt + \sigma(x, t) \, dz$. Also let $F(x(t), t)$ be at least a twice-differentiable function. Then the differential of $F(x, t)$ is

$$
\begin{align*}
    dF &= \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial t} \, dt + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \, (dx)^2 \\
    &= \left[ \frac{\partial F}{\partial x} \mu(x, t) + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2(x, t) \right] \, dt + \frac{\partial F}{\partial x} \sigma(x, t) \, dz
\end{align*}
$$

where the product $(dx)^2 = \sigma(x, t)^2 \, dt$. Hence, substituting in for $dx$ and $(dx)^2$, (20) can be rewritten:

$$
\begin{align*}
    dF &= \left[ \frac{\partial F}{\partial x} \mu(x, t) + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2(x, t) \right] \, dt + \frac{\partial F}{\partial x} \sigma(x, t) \, dz
\end{align*}
$$

(21)
Informal Proof

*Proof*: (See book for references to a formal proof, this is the intuition.)

Expand $F(x(t + \Delta t), t + \Delta t)$ in a Taylor series around $t$ and $x(t)$:

$$
F(x(t + \Delta t), t + \Delta t) = F(x(t), t) + \frac{\partial F}{\partial x} \Delta x + \frac{\partial F}{\partial t} \Delta t + \frac{1}{2} \left[ \frac{\partial^2 F}{\partial x^2} (\Delta x)^2 
+ 2 \frac{\partial^2 F}{\partial x \partial t} \Delta x \Delta t + \frac{\partial^2 F}{\partial t^2} (\Delta t)^2 \right] + H
$$

(22)

where $\Delta x \equiv x(t + \Delta t) - x(t)$ and $H$ represents terms with higher orders of $\Delta x$ and $\Delta t$. A discrete-time approximation of $\Delta x$ can be written as

$$
\Delta x = \mu(x, t) \Delta t + \sigma(x, t) \sqrt{\Delta t} \tilde{\epsilon}
$$

(23)
Informal Proof cont’d

Defining $\Delta F \equiv F(x(t + \Delta t), t + \Delta t) - F(x(t), t)$ and substituting (23) in for $\Delta x$, equation (22) can be rewritten as

$$
\Delta F = \frac{\partial F}{\partial x} \left( \mu(x, t) \Delta t + \sigma(x, t) \sqrt{\Delta t} \tilde{\epsilon} \right) + \frac{\partial F}{\partial t} \Delta t \\
+ \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \left( \mu(x, t) \Delta t + \sigma(x, t) \sqrt{\Delta t} \tilde{\epsilon} \right)^2 \\
+ \frac{\partial^2 F}{\partial x \partial t} \left( \mu(x, t) \Delta t + \sigma(x, t) \sqrt{\Delta t} \tilde{\epsilon} \right) \Delta t + \frac{1}{2} \frac{\partial^2 F}{\partial t^2} (\Delta t)^2 + H 
$$

Consider the limit as $\Delta t \to dt$ and $\Delta F \to dF$. Recall from (6) that $\sqrt{\Delta t} \tilde{\epsilon}$ becomes $dz$ and from (19) that $\left[ \sqrt{\Delta t} \tilde{\epsilon} \right] \left[ \sqrt{\Delta t} \tilde{\epsilon} \right]$ becomes $[dz(t)]^2 \to dt$. All terms of the form $dzdt \to 0$, and $dt^n \to 0$ as $\Delta t \to dt$ whenever $n > 1$. 
Informal Proof cont’d

\[(dx)^2 = (\mu(x, t) \, dt + \sigma(x, t) \, dz)^2 \quad (25)\]
\[= \mu(x, t)^2 \, (dt)^2 + 2\mu(x, t)\sigma(x, t) \, dtdz + \sigma(x, t)^2 \, (dz)^2 \]
\[= \sigma(x, t)^2 \, (dz)^2 = \sigma(x, t)^2 \, dt \]

So as \(\Delta t \to dt\), \(\sqrt{\Delta t} \tilde{\epsilon} \to dz\),

\[\Delta F = \frac{\partial F}{\partial x} \left( \mu(x, t) \Delta t + \sigma(x, t) \sqrt{\Delta t} \tilde{\epsilon} \right) + \frac{\partial F}{\partial t} \Delta t \]
\[+ \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \left( \mu(x, t) \Delta t + \sigma(x, t) \sqrt{\Delta t} \tilde{\epsilon} \right)^2 \]
\[+ \frac{\partial^2 F}{\partial x \partial t} \left( \mu(x, t) \Delta t + \sigma(x, t) \sqrt{\Delta t} \tilde{\epsilon} \right) \Delta t + \frac{1}{2} \frac{\partial^2 F}{\partial t^2} (\Delta t)^2 + H \]

becomes

\[dF = \left[ \frac{\partial F}{\partial x} \mu(x, t) + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2(x, t) \right] dt + \frac{\partial F}{\partial x} \sigma(x, t) \, dz \]
Geometric Brownian Motion

- Geometric Brownian motion is given by
  \[dx = \mu x \, dt + \sigma x \, dz\] (26)
  and is useful for modeling common stock prices since if \(x\) starts positive, it always remains positive (mean and variance are both proportional to its current value, \(x\)).

- Now consider \(F(x, t) = \ln(x)\), (e.g., \(dF = d(\ln x)\) is the rate of return). Applying Itô’s lemma, we have
  \[dF = d(\ln x) = \left[ \frac{\partial (\ln x)}{\partial x} \mu x + \frac{\partial (\ln x)}{\partial t} + \frac{1}{2} \frac{\partial^2 (\ln x)}{\partial x^2} (\sigma x)^2 \right] \, dt + \frac{\partial (\ln x)}{\partial x} \sigma x \, dz\]
  \[= \left[ \mu + 0 - \frac{1}{2} \sigma^2 \right] \, dt + \sigma \, dz\] (27)
Thus, $F = \ln x$ follows *arithmetic* Brownian motion. Since we know that

$$F(T) - F(0) \sim N \left( \left( \mu - \frac{1}{2} \sigma^2 \right) T, \sigma^2 T \right)$$  \hspace{1cm} (28)$$

then $x(t) = e^{F(t)}$ has a lognormal distribution over any discrete interval (by the definition of a lognormal random variable).

Hence, geometric Brownian motion is lognormally distributed over any time interval.
Backward Kolmogorov Equation

- In general, finding the discrete-time distribution of a variable that follows a diffusion is useful for:
  - computing its expected value
  - maximum likelihood estimation on discrete data

- Let $p(x, T; x_t, t)$ be the probability density function for diffusion $x$ at date $T$ given that it equals $x_t$ at date $t$, where $T \geq t$. Applying Itô’s lemma (assuming differentiability in $t$ and twice in $x_t$):

  \[
  dp = \left[ \frac{\partial p}{\partial x_t} \mu(x_t, t) + \frac{\partial p}{\partial t} + \frac{1}{2} \frac{\partial^2 p}{\partial x_t^2} \sigma^2(x_t, t) \right] dt + \frac{\partial p}{\partial x_t} \sigma(x_t, t) dz
  \]

- The expected change (i.e. drift) of $p$ should be zero.
Therefore,

$$\mu[x_t, t] \frac{\partial p}{\partial x_t} + \frac{\partial p}{\partial t} + \frac{1}{2} \sigma^2 (x_t, t) \frac{\partial^2 p}{\partial x_t^2} = 0 \quad (30)$$

Condition (30) is the backward Kolmogorov equation.

This partial differential equation for \( p(x, T; x_t, t) \) is solved subject to the boundary condition that when \( t \) becomes equal to \( T \), then \( x \) must equal \( x_t \) with probability 1.

Formally, this boundary condition is \( p(x, t; x_t, t) = \delta(x - x_t) \), where \( \delta(\cdot) \) is the Dirac delta function: \( \delta(0) = \infty, \delta(y) = 0 \ \forall \ y \neq 0 \), and \( \int_{-\infty}^{\infty} \delta(y) \, dy = 1 \).
Backward Kolmogorov Equation cont’d

- Example: if $\mu[x_t, t] = \mu x_t$, $\sigma^2(x_t, t) = \sigma^2 x_t^2$ (geometric Brownian motion), the Kolmogorov equation is

$$
\frac{1}{2} \sigma^2 x_t^2 \frac{\partial^2 p}{\partial x_t^2} + \mu x_t \frac{\partial p}{\partial x_t} + \frac{\partial p}{\partial t} = 0 \quad (31)
$$

- Substituting into (31), it can be verified that the solution is

$$
p(x, T, x_t, t) = \frac{1}{x \sqrt{2\pi \sigma^2 (T - t)}} \exp \left[ - \frac{(\ln x - \ln x_t - (\mu - \frac{1}{2} \sigma^2) (T - t))^2}{2\sigma^2 (T - t)} \right] \quad (32)
$$

which is the lognormal probability density function for the random variable $x \in (0, \infty)$. 

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Essentials of Diffusion Processes
Multivariate Diffusions and Itô’s Lemma

- Suppose there are $m$ diffusion processes
  \[
  dx_i = \mu_i \, dt + \sigma_i \, dz_i \quad i = 1, \ldots, m, \tag{33}
  \]
  and $dz_idz_j = \rho_{ij} \, dt$, where $\rho_{ij}$ is the correlation between Wiener process $dz_i$ and $dz_j$.
- Recall that $dz_idz_i = (dz_i)^2 = dt$. Now if $dz_{iu}$ is uncorrelated with $dz_i$, $dz_j$ can be written:
  \[
  dz_j = \rho_{ij} dz_i + \sqrt{1 - \rho_{ij}^2} dz_{iu} \tag{34}
  \]
  Then from this interpretation of $dz_j$, we have
  \[
  dz_j \, dz_j = \rho_{ij}^2 (dz_i)^2 + (1 - \rho_{ij}^2) (dz_{iu})^2 + 2 \rho_{ij} \sqrt{1 - \rho_{ij}^2} dz_i \, dz_{iu}
  = \rho_{ij}^2 \, dt + (1 - \rho_{ij}^2) \, dt + 0 \tag{35}
  = \, dt
  \]
Multivariate Itô’s Lemma

and

\[ dz_i dz_j = dz_i \left( \rho_{ij} dz_i + \sqrt{1 - \rho_{ij}^2} dz_{iu} \right) \]  \hspace{1cm} (36)

\[ = \rho_{ij} (dz_i)^2 + \sqrt{1 - \rho_{ij}^2} dz_i dz_{iu} \]

\[ = \rho_{ij} dt + 0 \]

- Thus, \( \rho_{ij} \) can be interpreted as the proportion of \( dz_j \) that is perfectly correlated with \( dz_i \).

- Let \( F(x_1, \ldots, x_m, t) \) be at least a twice-differentiable function. Then the differential of \( F(x_1, \ldots, x_m, t) \) is

\[ dF = \sum_{i=1}^{m} \frac{\partial F}{\partial x_i} dx_i + \frac{\partial F}{\partial t} dt + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial^2 F}{\partial x_i \partial x_j} dx_i dx_j \]  \hspace{1cm} (37)

where \( dx_i dx_j = \sigma_i \sigma_j \rho_{ij} dt \). Hence, (37) can be rewritten
Multivariate Itô’s Lemma cont’d

\[ dF = \left[ \sum_{i=1}^{m} \left( \frac{\partial F}{\partial x_i} \mu_i + \frac{1}{2} \frac{\partial^2 F}{\partial x_i^2} \sigma_i^2 \right) + \frac{\partial F}{\partial t} + \sum_{i=1}^{m} \sum_{j>i}^{m} \frac{\partial^2 F}{\partial x_i \partial x_j} \sigma_i \sigma_j \rho_{ij} \right] dt \]

\[ + \sum_{i=1}^{m} \frac{\partial F}{\partial x_i} \sigma_i \, dz_i \]  

(38)

- Equation (38) generalizes Itô’s lemma for a univariate diffusion, equation (21).
- Notably, the process followed by a function of several diffusion processes inherits each of the processes’ Brownian motions.
Summary

- Brownian motion is the foundation of diffusion processes and is a continuous-time limit of a discrete-time random walk.
- Itô’s lemma tells us how to find the process followed by a function of a diffusion process.
- The lemma can be used to derive the Kolmogorov equation, an important relation for finding the discrete-time distribution of a random variable that follows a diffusion process.
- The process followed by a function of several diffusions can be derived from a multivariate version of Itô’s lemma.