Arbitrage, Martingales, and Pricing Kernels

George Pennacchi

University of Illinois
Introduction

- A contingent claim’s price process can be transformed into a martingale process by
  1. Adjusting its Brownian motion by the market price of risk.
  2. Deflating by a riskless asset price.
- The claim’s value equals the expectation of the transformed process’s future payoff.
- We derive the continuous-time state price deflator that transforms actual probabilities into risk-neutral probabilities.
- Valuing a contingent claim might be simplified by deflating the contingent claim’s price by that of another risky asset.
- We consider applications: options on assets that pay a continuous dividend; the term structure of interest rates.
Arbitrage and Martingales

- Let $S$ be the value of a risky asset that follows a general scalar diffusion process

$$dS = \mu S dt + \sigma S dz$$  \hfill (1)

where both $\mu = \mu (S, t)$ and $\sigma = \sigma (S, t)$ may be functions of $S$ and $t$ and $dz$ is a Brownian motion.

- Itô’s lemma gives the process for a contingent claim’s price, $c(S, t)$:

$$dc = \mu_c c dt + \sigma_c c dz$$  \hfill (2)

where $\mu_c c = c_t + \mu Sc_S + \frac{1}{2} \sigma^2 S^2 c_{SS}$ and $\sigma_c c = \sigma Sc_S$, and the subscripts on $c$ denote partial derivatives.

- Consider a hedge portfolio of $-1$ units of the contingent claim and $c_S$ units of the risky asset.
Arbitrage and Martingales cont’d

- The value of this hedge portfolio, $H$, satisfies

$$H = -c + c_S S$$

(3)

and the change in its value over the next instant is

$$dH = -dc + c_S dS$$

(4)

$$= -\mu_c c dt - \sigma_c c dz + c_S \mu S dt + c_S \sigma S dz$$

$$= [c_S \mu S - \mu_c c] dt$$

- In the absence of arbitrage, the riskless portfolio change must be $H(t) r(t) dt$:

$$dH = [c_S \mu S - \mu_c c] dt = rH dt = r[-c + c_S S] dt$$

(5)
This no-arbitrage condition for \( dH \) implies:

\[
c_S \mu S - \mu_c c = r[-c + c_S S] \quad (6)
\]

Substituting \( \mu_c c = c_t + \mu S c_S + \frac{1}{2} \sigma^2 S^2 c_{SS} \) into (6) leads to the Black-Scholes equation:

\[
\frac{1}{2} \sigma^2 S^2 c_{SS} + rSc_S - rc + c_t = 0 \quad (7)
\]

However, a different interpretation of (6) results from substituting \( c_S = \frac{\sigma c c}{\sigma S} \) (from \( \sigma_c c = \sigma Sc_S \)):

\[
\frac{\mu - r}{\sigma} = \frac{\mu_c - r}{\sigma_c} \equiv \theta(t) \quad (8)
\]

No-arbitrage condition (8) requires a unique market price of risk, say \( \theta(t) \), so that \( \mu_c = r + \sigma_c \theta(t) \).
A Change in Probability

- Substituting for $\mu_c$ in (2) gives

$$dc = \mu_c c dt + \sigma_c c dz = \left[ rc + \theta \sigma_c c \right] dt + \sigma_c c dz$$  \hspace{1cm} (9)

- Next, consider a new process $\hat{z}_t = z_t + \int_0^t \theta (s) \, ds$, so that $d\hat{z}_t = dz_t + \theta (t) \, dt$.

- Then substituting $dz_t = d\hat{z}_t - \theta (t) \, dt$ in (9):

$$dc = \left[ rc + \theta \sigma_c c \right] dt + \sigma_c c [d\hat{z} - \theta dt] = r c dt + \sigma_c c d\hat{z}$$  \hspace{1cm} (10)

- If $\hat{z}_t$ were a Brownian motion, future values of $c$ generated by $d\hat{z}$ occur under the $Q$ or “risk-neutral” probability measure.

- The actual or “physical” distribution, $P$, is generated by the $dz$ Brownian motion.
Girsanov’s Theorem

- Let $dP_T$ be the instantaneous change in the cumulative distribution at date $T$ generated by $dz_t$ (the physical pdf).
- $dQ_T$ is the analogous risk-neutral pdf generated by $d\hat{z}_t$.
- Girsanov’s theorem says that at date $t < T$, the two probability densities satisfy

$$
dQ_T = \exp \left[ - \int_t^T \theta(u) \, dz - \frac{1}{2} \int_t^T \theta(u)^2 \, du \right] \, dP_T
$$

$$
= \left( \xi_T / \xi_t \right) \, dP_T
$$

(11)

where $\xi_t$ is a positive random process depending on $\theta(t)$ and $z_t$:

$$
\xi_T = \exp \left[ - \int_0^T \theta(u) \, dz - \frac{1}{2} \int_0^T \theta(u)^2 \, du \right]
$$

(12)
Girsanov’s Theorem cont’d

- Thus, multiplying the physical pdf at $T$ by $\xi_T/\xi_t$ leads to the risk-neutral pdf at $T$.

- Since $\xi_T/\xi_t > 0$, equation (11) implies that whenever $dP_T$ has positive probability, so does $dQ_T$, making them equivalent measures.

- Rearranging (11) gives the Radon-Nikodym derivative of $Q$ wrt $P$:
  
  $$\frac{dQ_T}{dP_T} = \frac{\xi_T}{\xi_t}$$  
  (13)

- Later we will relate this derivative to the continuous-time pricing kernel.
Money Market Deflator

- Let $B(t)$ be the value of an instantaneous-maturity riskless “money market fund” investment:

\[ \frac{dB}{B} = r(t)dt \quad (14) \]

- Note that $B(T) = B(t) e^{\int_t^T r(u)du}$ for any date $T \geq t$.

- Now define $C(t) \equiv c(t)/B(t)$ as the deflated price process for the contingent claim and use Itô’s lemma:

\[
dC = \frac{1}{B} dc - \frac{c}{B^2} dB = \frac{rc}{B} dt + \frac{\sigma_c c}{B} d\hat{z} - r \frac{c}{B} dt = \sigma_c C d\hat{z}
\]

since $dcdB = 0$ and we substitute for $dc$ from (10).
An implication of (15) is

\[ C(t) = \hat{E}_t [C(T)] \quad \forall T \geq t \]  (16)

where \( \hat{E}_t [\cdot] \) denotes the expectation operator under the probability measure generated by \( d\hat{Z} \).

Thus, \( C(t) \) is a martingale (random walk) process.

Note that (16) holds for any deflated non-dividend-paying contingent claim, including \( C = \frac{S}{B} \).

Later, we will consider assets that pay dividends.
Rewrite (16) in terms of the undeflated contingent claims price:

\[
c(t) = B(t) \hat{E}_t \left[ c(T) \frac{1}{B(T)} \right] \tag{17}
\]

\[
= \hat{E}_t \left[ e^{-\int_t^T r(u)du} c(T) \right]
\]

Equation (17) is the “Feynman-Kac” solution to the Black-Scholes PDE and does not require knowledge of \( \theta(t) \).

This is the continuous-time formulation of risk-neutral pricing: risk-neutral (or Q measure) expected payoffs are discounted by the risk-free rate.
Recall from the single- or multi-period consumption-portfolio choice problem with time-separable utility:

\[ c(t) = E_t \left[ m_{t,T} c(T) \right] \]

\[ = E_t \left[ \frac{M_T}{M_t} c(T) \right] \]

where date \( T \geq t \), \( m_{t,T} \equiv M_T / M_t \) and \( M_t = U_c(C_t, t) \).

Rewriting (18):

\[ c(t) M_t = E_t [c(T) M_T] \]

which says that the deflated price process, \( c(t) M_t \), is a martingale under \( P \) (not \( Q \)).
Assume that the state price deflator, $M_t$, follows a strictly positive diffusion process of the general form

$$dM_t = \mu_m dt + \sigma_m dz$$  \hspace{1cm} (20)

Define $c^m = cM$ and apply Itô’s lemma:

$$dc^m = c dM + M dc + (dc)(dM)$$  \hspace{1cm} (21)

$$= [c\mu_m + M\mu_c c + \sigma_c c\sigma_m] dt + [c\sigma_m + M\sigma_c c] dz$$

If $c^m = cM$ satisfies (19), that is, $c^m$ is a martingale, then its drift in (21) must be zero, implying

$$\mu_c = -\frac{\mu_m}{M} - \frac{\sigma_c \sigma_m}{M}$$  \hspace{1cm} (22)
Arbitrage and Pricing Kernels cont’d

Consider the case in which \( c \) is the instantaneously riskless investment \( B(t) \); that is, \( dc(t) = dB(t) = r(t)Bdt \) so that \( \sigma_c = 0 \) and \( \mu_c = r(t) \).

From (22), this requires

\[
\mu_c = r(t) = -\frac{\mu_m}{M}
\]  \hspace{1cm} (23)

Thus, the expected rate of change of the pricing kernel must equal minus the instantaneous risk-free interest rate.

Next, consider the general case where the asset \( c \) is risky, so that \( \sigma_c \neq 0 \). Using (22) and (23) together, we obtain

\[
\mu_c = r(t) - \frac{\sigma_c \sigma_m}{M}
\]  \hspace{1cm} (24)

or
Arbitrage and Pricing Kernels cont’d

\[ \frac{\mu_c - r}{\sigma_c} = -\frac{\sigma_m}{M} \]  \hspace{1cm} (25)

- Comparing (25) to (8), we see that

\[ -\frac{\sigma_m}{M} = \theta(t) \]  \hspace{1cm} (26)

- Thus, the no-arbitrage condition implies that the form of the pricing kernel must be

\[ dM/M = -r(t)\,dt - \theta(t)\,dz \]  \hspace{1cm} (27)
Define $m_t \equiv \ln M_t$ so that $dm = -[r + \frac{1}{2} \theta^2] dt - \theta dz$.

We can rewrite (18) as

$$ c(t) = E_t \left[ e^{-\int_t^T \left[r(u) + \frac{1}{2} \theta^2(u)\right] du - \int_t^T \theta(u) dz} \right] $$

Since the price under the money-market deflator ($Q$ measure) and the SDF ($P$ measure) must be the same, equating (17) and (28) implies

$$ \hat{E}_t \left[ e^{-\int_t^T r(u) du} c(T) \right] = E_t \left[ e^{mT - m_t} c(T) M_T / M_t \right] $$

$$ = E_t \left[ e^{-\int_t^T r(u) du} c(T) e^{-\int_t^T \frac{1}{2} \theta^2(u) du - \int_t^T \theta(u) dz} \right] $$
Substituting the definition of $\xi_T$ from (12) leads to

$$\hat{E}_t \left[ e^{-\int_t^T r(u)du} c(T) \right] = E_t \left[ e^{-\int_t^T r(u)du} c(T) (\xi_T/\xi_t) \right]$$

$$\hat{E}_t [C(T)] = E_t [C(T) (\xi_T/\xi_t)] \quad (30)$$

$$\int C(T) dQ_T = \int C(T) (\xi_T/\xi_t) dP_T$$

where $C(t) = c(t)/B(t)$. Thus, relating (29) to (30):

$$M_T/M_t = e^{-\int_t^T r(u)du} (\xi_T/\xi_t) \quad (31)$$

Hence, $M_T/M_t$ provides both discounting at the risk-free rate and transforming the probability distribution to the risk-neutral one via $\xi_T/\xi_t$. 
Multivariate Case

- Consider a multivariate extension where asset returns depend on an $n \times 1$ vector of independent Brownian motion processes, $\mathbf{dZ} = (dz_1 \ldots dz_n)'$ where $dz_i dz_j = 0$ for $i \neq j$.
- A contingent claim whose payoff depended on these asset returns has the price process

$$\frac{dc}{c} = \mu_c \, dt + \Sigma_c \, d\mathbf{Z} \quad (32)$$

where $\Sigma_c$ is a $1 \times n$ vector $\Sigma_c = (\sigma_{c1} \ldots \sigma_{cn})$.
- Let the corresponding $n \times 1$ vector of market prices of risks associated with each of the Brownian motions be $\Theta = (\theta_1 \ldots \theta_n)'$. 
Then the no-arbitrage condition (the multivariate equivalent of (8)) is

\[ \mu_c - r = \sum_c \Theta \]  \hspace{1cm} (33)

Equations (16) and (17) would still hold, and now the pricing kernel’s process would be given by

\[ \frac{dM}{M} = -r(t)\,dt - \Theta(t)'\,d\mathbf{Z} \]  \hspace{1cm} (34)
Consider an option written on the difference between two securities’ (stocks’) prices. The date $t$ price of stock 1, $S_1(t)$, follows the process

$$dS_1 / S_1 = \mu_1 dt + \sigma_1 dz_1$$

(35)

and the date $t$ price of stock 2, $S_2(t)$, follows the process

$$dS_2 / S_2 = \mu_2 dt + \sigma_2 dz_2$$

(36)

where $\sigma_1$ and $\sigma_2$ are assumed to be constants and $dz_1 dz_2 = \rho dt$.

Let $C(t)$ be the date $t$ price of a European option written on the difference between these two stocks’ prices.
Alternative Price Deflators cont’d

- At this option’s maturity date, \( T \), its value equals

\[
C(T) = \max [0, S_1(T) - S_2(T)]
\]  
(37)

- Now define \( c(t) = C(t) / S_2(t) \), \( s(t) = S_1(t) / S_2(t) \), and \( B(t) = S_2(t) / S_2(t) = 1 \) as the deflated price processes, where the prices of the option, stock 1, and stock 2 are all normalized by the price of stock 2.

- Under this normalized price system, the payoff (37) is

\[
c(T) = \max [0, s(T) - 1]
\]  
(38)

- Applying Itô’s lemma, the process for \( s(t) \) is

\[
ds/s = \mu_s dt + \sigma_s dz_3
\]  
(39)
Alternative Price Deflators cont’d

- Here \( \mu_s \equiv \mu_1 - \mu_2 + \sigma_2^2 - \rho \sigma_1 \sigma_2 \), \( \sigma_s dz_3 \equiv \sigma_1 dz_1 - \sigma_2 dz_2 \), and \( \sigma_s^2 = \sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2 \).

- Further, when prices are measured in terms of stock 2, the deflated price of stock 2 becomes the riskless asset with \( dB/B = 0 dt \) (the deflated price never changes).

- Using Itô’s lemma on \( c \),

\[
dc = \left[ c_s \mu_s s + c_t + \frac{1}{2} c_{ss} \sigma_s^2 s^2 \right] dt + c_s \sigma_s s dz_3 \quad (40)
\]

- The familiar Black-Scholes hedge portfolio can be created from the option and stock 1. The portfolio’s value is

\[
H = -c + c_s s \quad (41)
\]
Alternative Price Deflators cont’d

- The instantaneous change in value of the portfolio is

\[
dH = -dc + c_s ds
\]

\[
= - \left[ c_s \mu_s s + c_t + \frac{1}{2} c_{ss} \sigma_s^2 s^2 \right] dt - c_s \sigma_s s dz_3
\]

\[
+ c_s \mu_s s dt + c_s \sigma_s s dz_3
\]

\[
= - \left[ c_t + \frac{1}{2} c_{ss} \sigma_s^2 s^2 \right] dt
\]

which is riskless and must earn the riskless return \( dB/B = 0 \):

\[
dH = - \left[ c_t + \frac{1}{2} c_{ss} \sigma_s^2 s^2 \right] dt = 0 \tag{43}
\]

which implies

\[
c_t + \frac{1}{2} c_{ss} \sigma_s^2 s^2 = 0 \tag{44}
\]
This is the Black-Scholes PDE with the risk-free rate, \( r \), set to zero. With boundary condition (38), the solution is
\[
c(s, t) = s \, N(d_1) - N(d_2)
\] (45)

where
\[
d_1 = \frac{\ln(s(t)) + \frac{1}{2} \sigma_s^2 (T - t)}{\sigma_s \sqrt{T - t}}
\] (46)
\[
d_2 = d_1 - \sigma_s \sqrt{T - t}
\]

- Multiply by \( S_2 (t) \) to convert back to the undeflated price system:
\[
C(t) = S_1 \, N(d_1) - S_2 \, N(d_2)
\] (47)

- \( C(t) \) does not depend on \( r(t) \), so that this formula holds even for stochastic interest rates.
Continuous Dividends

- Let $S(t)$ be the date $t$ price per share of an asset that continuously pays a dividend of $\delta S(t)$ per unit time. Thus,

$$dS = (\mu - \delta) Sdt + \sigma Sdz$$

(48)

where $\sigma$ and $\delta$ are assumed to be constants.

- Note that the asset’s total rate of return is $dS/S + \delta dt = \mu dt + \sigma dz$, so that $\mu$ is its instantaneous expected rate of return.

- Consider a European call option written on this asset with exercise price of $X$ and maturity date of $T > t$, where we define $\tau \equiv T - t$.

- Let $r$ be the constant risk-free interest rate.
Based on (17), the date \( t \) price of this option is

\[
c(t) = \hat{E}_t [e^{-r\tau} c(T)]
\]

\[
= e^{-r\tau} \hat{E}_t [\max [S(T) - X, 0]]
\]

As in (10), convert from the physical measure generated by \( dz \) to the risk-neutral measure generated by \( d\hat{z} \), which removes the risk premium from the asset’s expected rate of return so that:

\[
dS = (r - \delta) Sdt + \sigma Sd\hat{z}
\]

Since \( r - \delta \) and \( \sigma \) are constants, \( S \) is a geometric Brownian motion process and is lognormally distributed under \( Q \).
Continuous Dividends cont’d

- Thus, the risk-neutral distribution of $\ln[S(T)]$ is normal:

  $$\ln[S(T)] \sim N\left(\ln[S(t)] + (r - \delta - \frac{1}{2}\sigma^2)\tau, \sigma^2\tau\right) \quad (51)$$

- Equation (49) can now be computed as

  $$c(t) = e^{-r\tau}\hat{E}_t\left[\max[S(T) - X, 0]\right] \quad (52)$$

  $$= e^{-r\tau} \int_{-\infty}^{\infty} (S(T) - X) g(S(T)) dS(T)$$

  where $g(S_T)$ is the lognormal probability density function.

- Consider the change in variable

  $$Y = \frac{\ln[S(T)/S(t)] - (r - \delta - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \quad (53)$$
Continuous Dividends cont’d

- $Y \sim N(0, 1)$ and allows (52) to be evaluated as

$$c = Se^{-\delta\tau}N(d_1) - Xe^{-r\tau}N(d_2)$$  \hspace{1cm} (54)

where

$$d_1 = \frac{\ln(S/X) + (r - \delta + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$  \hspace{1cm} (55)

- If contingent claims have more complex payoffs or the underlying asset has a more complex risk-neutral process, a numeric solution to $c(t) = \hat{E}_t[e^{-r\tau}c(S(T))]$ can be obtained, perhaps by Monte Carlo simulation.
Continuous Dividends cont’d

- Compared to an option written on an asset that pays no dividends, the non-dividend-paying asset’s price, $S(t)$, is replaced with the dividend-discounted price of the dividend-paying asset, $S(t) e^{-\delta \tau}$ (to keep the total expected rate of return at $r$).

- Thus, the risk-neutral expectation of $S(T)$ is

\[
\hat{E}_t [S(T)] = S(t) e^{(r-\delta) \tau} \\
= S(t) e^{-\delta \tau} e^{r \tau} = \overline{S}(t) e^{r \tau}
\]

where we define $\overline{S}(t) \equiv S(t) e^{-\delta \tau}$.
Foreign Currency Options

- Define $S(t)$ as the domestic currency value of a unit of foreign currency (spot exchange rate).
- Purchase of a foreign currency allows the owner to invest at the risk-free foreign currency interest rate, $r_f$.
- Thus the dividend yield will equal this foreign currency rate, $\delta = r_f$ and $\hat{E}_t[S(T)] = S(t) e^{(r-r_f)\tau}$.
- This expression is the no-arbitrage value of the date $t$ forward exchange rate having a time until maturity of $\tau$, that is, $F_{t,\tau} = Se^{(r-r_f)\tau}$.
- Therefore, a European option on foreign exchange is

$$c(t) = e^{-r\tau} \left[ F_{t,\tau} N(d_1) - XN(d_2) \right]$$

(57)

where $d_1 = \frac{\ln[F_{t,\tau}/X] + \sigma^2 \tau}{\sigma \sqrt{\tau}}$, and $d_2 = d_1 - \sigma \sqrt{\tau}$.
Options on Futures

- Consider an option written on a futures price $F_{t,t^*}$, the date $t$ futures price for a contract maturing at date $t^*$.

- The undiscounted profit (loss) earned by the long (short) party over the period from date $t$ to date $T \leq t^*$ is simply $F_{T,t^*} - F_{t,t^*}$.

- Like forward contracts, there is no initial cost for the parties who enter into a futures contract. Hence, in a risk-neutral world, their expected profits must be zero:

$$\hat{E}_t [F_{T,t^*} - F_{t,t^*}] = 0$$  \hspace{1cm} (58)

so under the $Q$ measure, the futures price is a martingale:

$$\hat{E}_t [F_{T,t^*}] = F_{t,t^*}$$  \hspace{1cm} (59)
Since an asset’s expected return under $Q$ must be $r$, a futures price is like the price of an asset with a dividend yield of $\delta = r$.

The value of a futures call option that matures in $\tau$ periods where $\tau \leq (t^* - t)$ is

$$c(t) = e^{-r\tau} [F_{t,t^*} N(d_1) - X N(d_2)]$$

(60)

where $d_1 = \frac{\ln[F_{t,t^*}/X] + \sigma^2 \tau}{\sigma \sqrt{\tau}}$, and $d_2 = d_1 - \sigma \sqrt{\tau}$.

Note that this is similar in form to an option on a foreign currency written in terms of the forward exchange rate.
Term Structure Revisited

- Let \( P(t, \tau) \) be the date \( t \) price of a default-free bond paying $1 at maturity \( T = t + \tau \).
- Interpreting \( c(T) = P(T, 0) = 1 \), equation (17) is
  \[
P(t, \tau) = \hat{E}_t \left[ e^{-\int_t^T r(u) \, du} \right]
  \quad (61)
\]
- We now rederive the Vasicek (1977) model using this equation where recall that the physical process for \( r(t) \) is
  \[
dr(t) = \alpha [\bar{r} - r(t)] \, dt + \sigma_r \, dz_r
  \quad (62)
\]
- Assuming, like before, that the market price of bond risk \( q \) is a constant,
  \[
  \mu_p(r, \tau) = r(t) + q \sigma_p(\tau)
  \quad (63)
  \]
  where \( \sigma_p(\tau) = -P_r \sigma_r / P \).
Thus, recall that the physical process for a bond’s price is

\[
dP(r, \tau) / P(r, \tau) = \mu_p(r, \tau) \, dt - \sigma_p(\tau) \, dz_r
\]

\[
= [r(t) + q \sigma_p(\tau)] \, dt - \sigma_p(\tau) \, dz_r
\]

(64)

Defining \( d\hat{z}_r = dz_r - qdt \), equation (64) becomes

\[
dP(t, \tau) / P(t, \tau) = [r(t) + q \sigma_p(\tau)] \, dt - \sigma_p(\tau) \, [d\hat{z}_r + qdt]
\]

\[
= r(t) \, dt - \sigma_p(\tau) \, d\hat{z}_r
\]

(65)

which is the risk-neutral process for the bond price since all bonds have the expected rate of return \( r \) under the \( Q \) measure.
Therefore, the process for $r(t)$ under the $Q$ measure is found by also substituting $d\hat{z}_r = dz_r - qdt$:

$$dr(t) = \alpha [\bar{r} - r(t)] dt + \sigma_r [d\hat{z}_r + qdt]$$

$$= \alpha \left[ \left( \bar{r} + \frac{q\sigma_r}{\alpha} \right) - r(t) \right] dt + \sigma_r d\hat{z}_r \quad (66)$$

which has the unconditional mean $\bar{r} + q\sigma_r/\alpha$.

Thus, when evaluating equation (61)

$$P(t, \tau) = \hat{E}_t \left[ \exp \left( - \int_t^T r(u) \, du \right) \right]$$

this expectation is computed assuming $r(t)$ follows the process in (66).

Doing so leads to the same solution given in the previous chapter, equation (9.41) in the text.
Summary

- Martingale pricing is a generalization of risk-neutral pricing that is applicable in complete markets.
- With dynamically complete markets, the continuous-time state price deflator has an expected growth rate equal to minus the risk-free rate and a standard deviation equal to the market price of risk.
- Contingent claims valuation often can be simplified by an appropriate normalization of asset prices, deflating either by the price of a riskless or risky asset.
- Martingale pricing can be applied to options written on assets paying continuous, proportional dividends, as well as default-free bonds.