Continuous-Time Consumption and Portfolio Choice

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Assuming that asset prices follow diffusion processes, we derive an individual’s continuous consumption and portfolio choices.

Asset demands reflect single-period mean-variance terms as well as components that hedge against changes in investment opportunities.

Consumption and portfolio choices can be solved using stochastic dynamic programming or, when markets are complete, a martingale technique.
Model Assumptions

- Let $x$ be a $k \times 1$ vector of state variables that affect the distribution of asset returns, where $r(x, t)$ is the date $t$ instantaneous-maturity risk-free rate and the date $t$ price of the $i^{th}$ risky asset, $S_i(t)$, follows the process

$$dS_i(t)/S_i(t) = \mu_i(x, t) \, dt + \sigma_i(x, t) \, dz_i \tag{1}$$

where $i = 1, \ldots, n$ and $(\sigma_i \, dz_i)(\sigma_j \, dz_j) = \sigma_{ij} \, dt$. The process (1) assumed the reinvestment of dividends.

- The $i^{th}$ state variable follow the process

$$dx_i = a_i(x, t) \, dt + b_i(x, t) \, d\zeta_i \tag{2}$$

where $i = 1, \ldots, k$. $d\zeta_i$ is a Brownian motion with $(b_i \, d\zeta_i)(b_j \, d\zeta_j) = b_{ij} \, dt$ and $(\sigma_i \, dz_i)(b_j \, d\zeta_j) = \phi_{ij} \, dt$. 
Model Assumptions cont’d

- Define $C_t$ as the individual’s date $t$ rate of consumption per unit time.
- Also, let $\omega_{i,t}$ be the proportion of total wealth at date $t$, $W_t$, allocated to risky asset $i$, $i = 1, ..., n$, so that

$$dW = \left[ \sum_{i=1}^{n} \omega_i dS_i / S_i + \left( 1 - \sum_{i=1}^{n} \omega_i \right) rdt \right] W - Cdt \tag{3}$$

$$= \sum_{i=1}^{n} \omega_i (\mu_i - r) W dt + (rW - C) dt + \sum_{i=1}^{n} \omega_i W \sigma_i dz_i$$

- Subject to (3), the individual solves:

$$\max_{C_s, \{ \omega_{i,s}, \forall s, i \}} E_t \left[ \int_t^T U(C_s, s) \, ds + B(W_T, T) \right] \tag{4}$$
Continuous-Time Dynamic Programming

Consider a simplified version of the problem in conditions (3) to (4) with only one choice and one state variable:

\[
\max_{\{c\}} E_t \left[ \int_t^T U(c_s, x_s) \, ds \right] \tag{5}
\]
subject to

\[
dx = a(x, c) \, dt + b(x, c) \, dz \tag{6}
\]

where \(c_t\) is a control (e.g. consumption) and \(x_t\) is a state (e.g. wealth). Define the indirect utility function, \(J(x_t, t)\):

\[
J(x_t, t) = \max_{\{c\}} E_t \left[ \int_t^T U(c_s, x_s) \, ds \right] \tag{7}
\]

\[
= \max_{\{c\}} E_t \left[ \int_t^{t+\Delta t} U(c_s, x_s) \, ds + \int_{t+\Delta t}^T U(c_s, x_s) \, ds \right]
\]
Apply Bellman's Principle of Optimality:

\[ J(x_t, t) = \max_{c} E_t \left[ \int_t^{t+\Delta t} U(c_s, x_s) \, ds + \max_{\{c\}} \left( E_{t+\Delta t} \left\{ \int_{t+\Delta t}^{T} U(c_s, x_s) \, ds \right\} \right) \right] \]

\[ = \max_{\{c\}} E_t \left[ \int_t^{t+\Delta t} U(c_s, x_s) \, ds + J(x_{t+\Delta t}, t + \Delta t) \right] \]  \hspace{1cm} (8)

For \( \Delta t \) small, approximate the first integral as \( U(c_t, x_t) \, \Delta t \) and expand \( J(x_{t+\Delta t}, t + \Delta t) \) around \( x_t \) and \( t \) in a Taylor series:

\[ J(x_t, t) = \max_{\{c\}} E_t \left[ U(c_t, x_t) \Delta t + J(x_t, t) + J_x \Delta x + J_t \Delta t \right. \]

\[ + \frac{1}{2} J_{xx}(\Delta x)^2 + J_{xt}(\Delta x)(\Delta t) + \frac{1}{2} J_{tt}(\Delta t)^2 + o(\Delta t) \left. \right] \]  \hspace{1cm} (9)

where \( o(\Delta t) \) represents higher-order terms.
The state variable’s diffusion process (6) is approximated

\[ \Delta x \approx a(x, c) \Delta t + b(x, c) \Delta z + o(\Delta t) \quad (10) \]

where \( \Delta z = \sqrt{\Delta t} \tilde{\varepsilon} \) and \( \tilde{\varepsilon} \sim N(0, 1) \). Substituting (10) into (9), and subtracting \( J(x_t, t) \) from both sides,

\[ 0 = \max_{c} E_t [U(c_t, x_t) \Delta t + \Delta J + o(\Delta t)] \quad (11) \]

where

\[ \Delta J = \left[ J_t + J_x a + \frac{1}{2} J_{xx} b^2 \right] \Delta t + J_x b \Delta z \quad (12) \]

This is just a discrete-time version of Itô’s lemma. In equation (11), \( E_t [J_x b \Delta z] = 0 \). Divide both sides of (11) by \( \Delta t \).
Continuous-Time Dynamic Programming cont’d

- We can take the limit as $\Delta t \to 0$:

  $$0 = \max_{\{c\}} \left[ U(c_t, x_t) + J_t + J_x a + \frac{1}{2} J_{xx} b^2 \right]$$  \hspace{1cm} (13)

- Equation (13) is the stochastic, continuous-time Bellman equation and can be rewritten as

  $$0 = \max_{\{c\}} [U(c_t, x_t) + L[J]]$$  \hspace{1cm} (14)

where $L[\cdot]$ is the Dynkin operator; that is, the “drift” term (expected change per unit of time) in $dJ(x, t)$ obtained from applying Itô’s lemma to $J$. 
Solving the *Real* Continuous-Time Problem

- Returning to the consumption - portfolio choice problem, define the indirect utility-of-wealth $J(W, x, t)$:

$$J(W, x, t) = \max_{C_s, \{\omega_{i,s}\}, \forall s, i} \mathbb{E}_t \left[ \int_t^T U(C_s, s) \, ds + B(W_T, T) \right]$$  \hspace{1cm} (15)

- In this problem, consumption, $C_t$, and portfolio weights, $\{\omega_{i,t}\}, i = 1, \ldots, n$ are the control variables.

- Wealth, $W_t$, and the variables affecting the distribution of asset returns, $x_{i,t}, i = 1, \ldots, k$ are the state variables that evolve according to (1) and (2), respectively.
Solving the Continuous-Time Problem

Thus, the Dynkin operator in terms of $W$ and $x$ is

$$L[J] = \frac{\partial J}{\partial t} + \left[ \sum_{i=1}^{n} \omega_i (\mu_i - r) W + (rW - C) \right] \frac{\partial J}{\partial W} + \sum_{i=1}^{k} a_i \frac{\partial J}{\partial x_i}$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \omega_i \omega_j W^2 \frac{\partial^2 J}{\partial W^2} + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} b_{ij} \frac{\partial^2 J}{\partial x_i \partial x_j}$$

$$+ \sum_{j=1}^{k} \sum_{i=1}^{n} W \omega_i \phi_{ij} \frac{\partial^2 J}{\partial W \partial x_j}$$

(16)

From equation (14) we have

$$0 = \max_{C_t, \{\omega_i, t\}} [U(C_t, t) + L[J]]$$

(17)

We obtain first-order conditions wrt $C_t$ and $\omega_{i,t}$:
Solving the Continuous-Time Problem cont’d

\[ 0 = \frac{\partial U(C^*, t)}{\partial C} - \frac{\partial J(W, x, t)}{\partial W} \]  \hspace{1cm} (18)

\[ 0 = W \frac{\partial J}{\partial W} (\mu_i - r) + W^2 \frac{\partial^2 J}{\partial W^2} \sum_{j=1}^{n} \sigma_{ij} \omega_j^* + W \sum_{j=1}^{k} \phi_{ij} \frac{\partial^2 J}{\partial x_j \partial W}, \]  \hspace{1cm} (19)

where \( i = 1, \ldots, n \).

- Equation (18) is the envelope condition while equation (19) has the discrete-time analog

\[ E_t [R_{i,t} J_W (W_{t+1}, t+1)] = R_{f,t} E_t [J_W (W_{t+1}, t+1)], \quad i = 1, \ldots, n \]
Solving the Continuous-Time Problem cont’d

- Define the inverse marginal utility function $G = [\partial U / \partial C]^{-1}$ and let $J_W$ be shorthand for $\partial J / \partial W$. Condition (18) becomes

$$C^* = G (J_W, t) \quad (20)$$

- Denote $\Omega \equiv [\sigma_{ij}]$ as the $n \times n$ instantaneous covariance matrix whose $i, j^{th}$ element is $\sigma_{ij}$, and denote $\nu_{ij}$ as the $i, j^{th}$ element of $\Omega^{-1} \equiv [\nu_{ij}]$.

- Then the solution to (19) can be written as

$$\omega_i^* = -\frac{J_W}{J_{WW} W} \sum_{j=1}^{n} \nu_{ij} (\mu_j - r) - \sum_{m=1}^{k} \sum_{j=1}^{n} \frac{J_{Wmx}}{J_{WW} W} \phi_{im} \nu_{ij}, \quad i = 1, \ldots, n \quad (21)$$

- $\omega_i^*$ in (21) depends on $-J_W / (J_{WW} W)$ which is the inverse of relative risk aversion for lifetime utility of wealth.
Assuming specific functions for $U$ and the $\mu_i$’s, $\sigma_{ij}$’s, and $\phi_{ij}$’s, equations (20) and (21) can be solved in terms of the state variables $W$, $x$, and $J_W$, $J_{WW}$, and $J_{WX_i}$.

Substituting $C^*$ and the $\omega^*_i$ back into equation (17) leads to a nonlinear partial differential equation (PDE) for $J$ that can be solved subject to $J(W_T, x_T, T) = B(W_T, T)$.

In turn, solutions for $C^*_t$ and the $\omega^*_{i,t}$ in terms of only $W_t$, and $x_t$ then result from (20) and (21).

If all of the $\mu_i$’s (including $r$) and $\sigma_i$’s are constants, asset returns are lognormally distributed and there is a constant investment opportunity set.

In this case the only state variable is $W$, and the optimal portfolio weights in (21) simplify to...
Constant Investment Opportunities

\[
\omega_i^* = -\frac{J_W}{J_{WW}W} \sum_{j=1}^{n} \nu_{ij}(\mu_j - r), \quad i = 1, \ldots, n \tag{22}
\]

- Plugging (20) and (22) back into the optimality equation (17), and using the fact that \([\nu_{ij}] \equiv \Omega^{-1}\), we have

\[
0 = U(G, t) + J_t + \left[ \sum_{i=1}^{n} \omega_i(\mu_i - r)W + rW - C \right] J_W + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \omega_i \omega_j W^2 J_{WW}
\]

\[
= U(G) + J_t + J_W(rW - G) - \frac{J_W^2}{J_{WW}} \sum_{i=1}^{n} \sum_{j=1}^{n} \nu_{ij}(\mu_i - r)(\mu_j - r)
\]

\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \omega_i \omega_j W^2 \frac{\partial^2 J}{\partial W^2}
\]

\[
= U(G) + J_t + J_W(rW - G) - \frac{J_W^2}{2J_{WW}} \sum_{i=1}^{n} \sum_{j=1}^{n} \nu_{ij}(\mu_i - r)(\mu_j - r) \tag{23}
\]
This equation can be solved for $J$ and, in turn, $C^*$ and $\omega_i^*$ after specifying $U$.

In any case, since $\nu_{ij}$, $\mu_j$, and $r$ are constants, the proportion of each risky asset that is optimally held will be proportional to $-JW/(JWWW)$ which is common across all assets.

Consequently, the proportion of wealth in risky asset $i$ to risky asset $k$ is a constant:

$$\frac{\omega_i^*}{\omega_k^*} = \frac{\sum_{j=1}^{n} \nu_{ij}(\mu_j - r)}{\sum_{j=1}^{n} \nu_{kj}(\mu_j - r)}$$

(24)
Therefore, the proportion of risky asset $k$ to all risky assets is

$$\delta_k = \frac{\omega_k^*}{\sum_{i=1}^{n} \omega_i^*} = \frac{\sum_{j=1}^{n} \nu_{kj}(\mu_j - r)}{\sum_{i=1}^{n} \sum_{j=1}^{n} \nu_{ij}(\mu_j - r)}$$  \hspace{1cm} (25)$$

Since all individuals regardless of $U$ will hold $r$ and the constant-proportion portfolio of risky assets defined by $\delta_k$, we obtain a two-fund separation result: all individuals’ optimal portfolios consists of the risk-free asset paying rate of return $r$ and a single risky asset portfolio having the following expected rate of return, $\mu$, and variance, $\sigma^2$: 
Two-Fund Separation

\[ \mu \equiv \sum_{i=1}^{n} \delta_i \mu_i \]  
\[ \sigma^2 \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i \delta_j \sigma_{ij} \]  

Indeed, recalling the single-period mean-variance portfolio weights, the \( i^{th} \) element of (2.42) can be written as \( w_i^* = \lambda \sum_{j=1}^{n} \nu_{ij} (\overline{R}_j - R_f) \), which equals (22) when \( \lambda = -J_W / (J_{WW} W) \).

Hence, we obtain mean-variance portfolio weights with lognormally-distributed asset returns since the asset return diffusions are \textit{locally} normal.
HARA Utility and Constant Investment Opportunities

- Analytic solutions to the constant investment opportunity problem exist with Hyperbolic Absolute Risk Aversion utility:

\[ U(C, t) = e^{-\rho t} \frac{1 - \gamma}{\gamma} \left( \frac{\alpha C}{1 - \gamma} + \beta \right)^\gamma \]  \tag{27}

- Optimal consumption in equation (20) is

\[ C^* = \frac{1 - \gamma}{\alpha} \left[ \frac{e^{\rho t} J_W}{\alpha} \right]^{\frac{1}{\gamma - 1}} - \frac{(1 - \gamma) \beta}{\alpha} \]  \tag{28}

and using (22) and (26), the risky-asset portfolio weights are

\[ \omega^* = - \frac{J_W}{J_{WW} W} \frac{\mu - r}{\sigma^2} \]  \tag{29}
HARA Utility and Constant Investment Opportunities

- Simplify equation (23) to obtain

\[ 0 = \frac{(1 - \gamma)^2}{\gamma} e^{-\rho t} \left[ \frac{e^{\rho t} J_W}{\alpha} \right]^\frac{\gamma}{\gamma - 1} + J_t + \left( \frac{(1 - \gamma)\beta}{\alpha} + rW \right) J_W - \frac{J_W^2}{J_{WW}} \frac{(\mu - r)^2}{2\sigma^2} \] (30)

- Merton (1971) solves this PDE subject to

\[ J(W, T) = B(W_T, T) = 0, \] and shows (28) and (29) then take the form

\[ C_t^* = aW_t + b \] (31)

and

\[ \omega_t^* = g + \frac{h}{W_t} \] (32)
CRRA and Constant Investment Opportunities

- Here $a$, $b$, $g$, and $h$ are, at most, functions of time.
- For the special case of constant relative risk aversion where $U(C, t) = e^{-\rho t} C^{\gamma}/\gamma$, the solution is

$$J(W, t) = e^{-\rho t} \left[ \frac{1 - e^{-\kappa(T-t)}}{\kappa} \right]^{1-\gamma} W^{\gamma}/\gamma$$  \hspace{1cm} (33)$$

$$C_t^* = \frac{\kappa}{1 - e^{-\kappa(T-t)}} W_t$$  \hspace{1cm} (34)$$

and

$$\omega^* = \frac{\mu - r}{(1 - \gamma)\sigma^2}$$  \hspace{1cm} (35)$$

where $\kappa \equiv \frac{\gamma}{1-\gamma} \left[ \frac{\rho}{\gamma} - r - \frac{(\mu-r)^2}{2(1-\gamma)\sigma^2} \right]$.  

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Continuous-Time Consumption and Portfolio Choice
Implications of Continuous-Time Decisions

- When the individual’s planning horizon is infinite, $T \to \infty$, a solution exists only if $\kappa > 0$.
- In this case with $T \to \infty$, $C_t^* = \kappa W_t$.
- Although we obtain the Markowitz result in continuous time, it is not the same result as in discrete time.
- For example, a CRRA individual facing normally distributed returns and discrete-time portfolio rebalancing will choose to put all wealth in the risk-free asset.
- In contrast, this individual facing lognormally-distributed returns and continuous portfolio rebalancing chooses $\omega^* = (\mu - r) / [(1 - \gamma)\sigma^2]$, which is independent of the time horizon.
Consider the effects of changing investment opportunities by simply assuming a single state variable so that $k = 1$ and $x$ is a scalar that follows the process

$$dx = a(x, t) \, dt + b(x, t) \, d\zeta$$

(36)

where $b \, d\zeta \sigma_i \, dz_i = \phi_i \, dt$.

The optimal portfolio weights in (21) are

$$\omega_i^* = -\frac{J_W}{WJ_{WW}} \sum_{j=1}^{n} \nu_{ij} (\mu_j - r) - \frac{J_{Wx}}{WJ_{WW}} \sum_{j=1}^{n} \nu_{ij} \phi_j, \quad i = 1, \ldots, n$$

(37)
Portfolio Weights with Changing Investment Opportunities

- Written in matrix form, equation (37) is

\[
\omega^* = \frac{A}{W} \Omega^{-1} (\mu - re) + \frac{H}{W} \Omega^{-1} \phi
\]  

(38)

where \( \omega^* = (\omega_1^* \ldots \omega_n^*)' \) is the \( n \times 1 \) vector of portfolio weights for the \( n \) risky assets; \( \mu = (\mu_1 \ldots \mu_n)' \) is the \( n \times 1 \) vector of these assets’ expected rates of return; \( e \) is an \( n \)-dimensional vector of ones, \( \phi = (\phi_1, \ldots, \phi_n)' \), \( A = -J_{W} W \Omega \), and \( H = -J_{Wx} W \Omega \).

- \( A \) and \( H \) will, in general, differ from one individual to another, depending on the form of the particular individual’s utility function and level of wealth.
Three Fund Theorem

- Thus, unlike in the constant investment opportunity set case (where $J_{Wx} = H = 0$), $\omega_i^*/\omega_j^*$ is not the same for all investors.
- A two mutual fund theorem does not hold, but with one state variable, $x$, a three fund theorem does hold.
- Investors will be satisfied choosing between
  1. A fund holding the risk-free asset.
  2. A mean-variance efficient fund with weights $\Omega^{-1} (\mu - re)$.
  3. A fund with weights $\Omega^{-1} \phi$ that best hedges against changing investment opportunities.
Portfolio Demands

- Recall $J_W = U_C$, which allows us to write $J_{WW} = U_{CC} \partial C / \partial W$.

- Therefore, $A$ can be rewritten as

$$A = - \frac{U_C}{U_{CC} (\partial C / \partial W)} > 0 \quad (39)$$

by the concavity of $U$. Also, since $J_{Wx} = U_{CC} \partial C / \partial x$,

$$H = - \frac{\partial C / \partial x}{\partial C / \partial W} \geq 0 \quad (40)$$

- $A$ is proportional to the reciprocal of the individual’s absolute risk aversion, so the smaller is $A$, the smaller in magnitude is the individual’s demand for any risky asset.
Hedging Demand

- An unfavorable shift in investment opportunities is defined as a change in x such that consumption falls, that is, an increase in x if \( \frac{\partial C}{\partial x} < 0 \) and a decrease in x if \( \frac{\partial C}{\partial x} > 0 \).
- For example, suppose \( \Omega \) is a diagonal matrix, so that \( v_{ij} = 0 \) for \( i \neq j \) and \( v_{ii} = 1/\sigma_{ii} > 0 \), and also assume that \( \phi_i \neq 0 \). In this case, the hedging demand for risky asset \( i \) in (38) is

\[
H v_{ii} \phi_i = -\frac{\partial C}{\partial x} \frac{\partial C}{\partial W} v_{ii} \phi_i > 0 \text{ iff } \frac{\partial C}{\partial x} \phi_i < 0
\]  

(41)

- Thus, if \( \frac{\partial C}{\partial x} < 0 \) and if \( x \) and asset \( i \) are positively correlated (\( \phi_i > 0 \)), then there is a positive hedging demand for asset \( i \); that is, \( H v_{ii} \phi_i > 0 \) and asset \( i \) is held in greater amounts than what would be predicted based on a simple single-period mean-variance analysis.
Changing Interest Rate Example

- Let $r = x$ and $\mu = re + p = xe + p$ where $p$ is a vector of risk premia for the risky assets.

- Thus, an increase in the risk-free rate $r$ indicates an improvement in investment opportunities.

- Recall that in a simple certainty model with constant relative-risk-aversion utility, the elasticity of intertemporal substitution is given by $\epsilon = 1 / (1 - \gamma)$.

- When $\epsilon < 1$, implying that $\gamma < 0$, an increase in the risk-free rate leads to greater current consumption consistent with equation (34) where, for the infinite horizon case $C_t = \frac{\gamma}{1 - \gamma} \left[ \frac{p}{\gamma} - r - \frac{(\mu - r)^2}{2(1 - \gamma)\sigma^2} \right] W_t = \frac{\gamma}{1 - \gamma} \left[ \frac{p}{\gamma} - r - \frac{p^2}{2(1 - \gamma)\sigma^2} \right] W_t$, so that $\partial C_t / \partial r = -\gamma W_t / (1 - \gamma)$. 
Asset Allocation Puzzle

- Given empirical evidence that risk aversion is greater than log ($\gamma < 0$), the intuition from these simple models would be that $\partial C_t / \partial r > 0$ and is increasing in risk aversion.
- From equation (41) we have
  \[ Hv_{ii} \phi_i = - \frac{\partial C}{\partial W} \frac{\partial C}{\partial r} v_{ii} \phi_i > 0 \text{ iff } \frac{\partial C}{\partial r} \phi_i < 0 \] 
  (42)
- Thus, there is a positive hedging demand for an asset that is negatively correlated with changes in the interest rate, $r$.
- An obvious candidate asset is a long-maturity bond.
- This insight can explain why financial planners recommend both greater cash and a greater bonds-to-stocks mix for more risk-averse investors (the Asset Allocation Puzzle of Canner, Mankiw, and Weil AER 1997).
Log Utility

- Logarithmic utility is one of the few cases in which analytical solutions are possible for consumption and portfolio choices when investment opportunities are changing.
- Suppose $U(C_s, s) = e^{-\rho s} \ln(C_s)$ and $B(W_T, T) = e^{-\rho T} \ln(W_T)$.
- Consider a trial solution to (17) for the indirect utility function of the form $J(W, x, t) = d(t) U(W_t, t) + F(x, t)$

\[= d(t) e^{-\rho t} \ln(W_t) + F(x, t).\]
- If so, then (20) is

\[C_t^* = \frac{W_t}{d(t)} \quad (43)\]

and (37) simplifies to

\[\omega_i^* = \sum_{j=1}^{n} \nu_{ij} (\mu_j - r) \quad (44)\]
Log Utility

Substituting $C_t^*$ and $\omega_i^*$ into the Bellman equation (17):

\[
0 = U(C_t^*, t) + J_t + J_W [rW_t - C_t^*] + a(x, t) J_x \\
+ \frac{1}{2} b(x, t)^2 J_{xx} - \frac{J_W^2}{2J_{WW}} \sum_{i=1}^{n} \sum_{j=1}^{n} \nu_{ij} (\mu_j - r) (\mu_i - r) \\
= e^{-\rho t} \ln \left[ \frac{W_t}{d(t)} \right] + e^{-\rho t} \left[ \frac{\partial d(t)}{\partial t} - \rho d(t) \right] \ln [W_t] + F_t \\
+ e^{-\rho t} d(t) r - e^{-\rho t} + a(x, t) F_x + \frac{1}{2} b(x, t)^2 F_{xx} \\
+ \frac{d(t) e^{-\rho t}}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \nu_{ij} (\mu_j - r) (\mu_i - r)
\]

(45)
Simplifying, the equation becomes

$$0 = -\ln[d(t)] + \left[1 + \frac{\partial d(t)}{\partial t} - \rho d(t)\right] \ln[W_t] + e^{\rho t} F_t$$

$$+ d(t) r - 1 + a(x, t) e^{\rho t} F_x + \frac{1}{2} b(x, t)^2 e^{\rho t} F_{xx}$$

$$+ \frac{d(t)}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} (\mu_j - r) (\mu_i - r)$$

(46)
Since a solution must hold for all values of wealth, we must have

\[
\frac{\partial d(t)}{\partial t} - \rho d(t) + 1 = 0 \quad (47)
\]

subject to the boundary condition \(d(T) = 1\).

The solution to this first-order ordinary differential equation is

\[
d(t) = \frac{1}{\rho} \left[ 1 - (1 - \rho) e^{-\rho(T-t)} \right] \quad (48)
\]
The complete solution to (46) is then to solve

\[
0 = -\ln [d(t)] + e^{\rho t} F_t + d(t) r - 1 + a(x, t) e^{\rho t} F_x + \frac{1}{2} b(x, t)^2 e^{\rho t} F_{xx} + \frac{d(t)}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \nu_{ij} (\mu_j - r)(\mu_i - r)
\]

subject to the boundary condition \( F(x, T) = 0 \).

The solution depends on how \( r \), the \( \mu_i \)'s, and \( \Omega \) are assumed to depend on the state variable \( x \).

However, these relationships influence only the level of indirect utility via \( F(x, t) \) and do not affect \( C_t^* \) and \( \omega_i^* \).
Log Utility

- Substituting (48) into (43), consumption is

\[ C_t = \frac{\rho}{1 - (1 - \rho) e^{-\rho(T-t)}} W_t \]  \hspace{1cm} (50)

which is comparable to our earlier discrete-time problem.

- The log utility investor behaves myopically by having no desire to hedge against changes in investment opportunities, though the portfolio weights \( \omega_i^* = \sum_{j=1}^{n} v_{ij} (\mu_j - r) \) will change over time as \( v_{ij}, \mu_j, \) and \( r \) change.
The Martingale Approach

- Modify process (1) to write the return on risky $i$ as

$$dS_i/S_i = \mu_i dt + \Sigma_i d\mathbf{Z}, \quad i = 1, \ldots, n$$  \hspace{1cm} (51)

where $\Sigma_i = (\sigma_{i1} \ldots \sigma_{in})$ is a $1 \times n$ vector of volatility terms and $d\mathbf{Z} = (dz_1 \ldots dz_n)'$ is an $n \times 1$ vector of independent Brownian motions.

- $\mu_i$, $\Sigma_i$, and $r(t)$ may be functions of state variables driven by the Brownian motion elements of $d\mathbf{Z}$.

- If $\Sigma$ is the $n \times n$ matrix whose $i^{th}$ row equals $\Sigma_i$, then the covariance matrix of the assets' returns is $\Omega \equiv \Sigma \Sigma'$. 

Complete Market Assumptions

- Importantly, we now assume that uncertain changes in the means and covariances of the asset return processes in (51) are driven only by the vector $dZ$.

- Equivalently, each state variable, say $x_i$ as represented in (2), has a Brownian motion process, $d\zeta_i$, that is a linear function of $dZ$.

- Thus, changes in investment opportunities can be *perfectly* hedged by the $n$ assets so that markets are dynamically complete.
Pricing Kernel

Using a Black-Scholes hedging argument and the absence of arbitrage, we showed that a stochastic discount factor exists and follows the process

\[ \frac{dM}{M} = -rdt - \Theta(t)'dZ \]  \hspace{1cm} (52)

where \( \Theta = (\theta_1 \ldots \theta_n)' \) is an \( n \times 1 \) vector of market prices of risks associated with each Brownian motion and

\[ \mu_i - r = \sum_i \Theta, \quad i = 1, \ldots, n \]  \hspace{1cm} (53)
Optimal Consumption Plan

- Note that the individual’s wealth equals the expected discounted value of the dividends (consumption) that it pays over the individual’s planning horizon plus discounted terminal wealth

$$W_t = E_t \left[ \int_t^T \frac{M_s}{M_t} C_s ds + \frac{M_T}{M_t} W_T \right] \quad \text{(54)}$$

- Equation (54) can be interpreted as an intertemporal budget constraint.
Static Optimization Problem

The choice of consumption and terminal wealth can be transformed into a static optimization problem by the following Lagrange multiplier problem:

\[
\max_{C_s \forall s \in [t, T], W_T} E_t \left[ \int_t^T U(C_s, s) \, ds + B(W_T, T) \right] + \lambda \left( M_t W_t - E_t \left[ \int_t^T M_s C_s ds + M_T W_T \right] \right)
\]

(55)

Later, we address the portfolio choice problem that would implement the consumption plan.
First Order Conditions

- Treating the integrals in (55) as summations over infinite points in time, the first-order conditions for optimal consumption at each date and for terminal wealth are

\[
\frac{\partial U(C_s,s)}{\partial C_s} = \lambda M_s, \quad \forall s \in [t, T]
\]  

(56)

\[
\frac{\partial B(W_T,T)}{\partial W_T} = \lambda M_T
\]  

(57)

- Define the inverse functions \( G = [\partial U/\partial C]^{-1} \) and \( G_B = [\partial B/\partial W]^{-1} \):

\[
C_s^* = G(\lambda M_s, s), \quad \forall s \in [t, T]
\]  

(58)

\[
W_T^* = G_B(\lambda M_T, T)
\]  

(59)
Determining the Lagrange multiplier

- Substitute (58) and (59) into (54) to obtain

\[
W_t = E_t \left[ \int_t^T \frac{M_s}{M_t} G(\lambda M_s, s) ds + \frac{M_T}{M_t} G_B(\lambda M_T, T) \right] \tag{60}
\]

- Given the initial wealth, \( W_t \), the distribution of \( M_s \) from (52), and the forms of the utility and bequest functions (which determine \( G \) and \( G_B \)), the expectation in equation (60) can be calculated to determine \( \lambda \) as a function of \( W_t, M_t \), and any date \( t \) state variables.
Alternative Solution for the Multiplier

Since $W_t$ represents a contingent claim that pays a dividend equal to consumption, it must satisfy a particular Black-Scholes-Merton partial differential equation (PDE).

For example, assume that $\mu_i$, $\Sigma_i$, and $r(t)$ are functions of a single state variable, say, $x_t$, that follows the process

$$dx = a(x, t) \, dt + B(x, t)' \, dZ$$

(61)

where $B(x, t) = (B_1 \ldots B_n)'$ is an $n \times 1$ vector of volatilities multiplying the Brownian motion components of $dZ$.

Based on (60) and the Markov nature of $M_t$ in (52) and $x_t$ in (61), the date $t$ value of optimally invested wealth is a function of $M_t$ and $x_t$ and the individual’s time horizon, $W(M_t, x_t, t)$. 
Wealth Process

- By Itô’s lemma, $W(M_t, x_t, t)$ follows the process

\[
\begin{align*}
    dW &= W_M dM + W_x dx + \frac{\partial W}{\partial t} dt + \frac{1}{2} W_{MM} (dM)^2 \\
    &\quad + W_{Mx} (dM) (dx) + \frac{1}{2} W_{xx} (dx)^2 \\
    &= \mu_W dt + \Sigma_W dZ
\end{align*}
\]

where

\[
\begin{align*}
    \mu_W &\equiv -rMW_M + aW_x + \frac{\partial W}{\partial t} + \frac{1}{2} \Theta' \Theta M^2 W_{MM} \\
    &\quad - \Theta' B MW_{Mx} + \frac{1}{2} B'B W_{xx} \\
    \Sigma_W &\equiv -W_M M \Theta + W_x B
\end{align*}
\]
No Arbitrage Condition for Wealth

- The expected return on wealth must earn the instantaneous risk-free rate plus its risk premium:

\[ \mu_W + G (\lambda M_t, t) = rW_t + \Sigma'_W \Theta \]  

(65)

- Substituting in for \( \mu_W \) and \( \Sigma'_W \) leads to the PDE

\[
0 = \Theta' \Theta M^2 \frac{W_{MM}}{2} - \Theta' \mathbf{B} M W_{Mx} + \mathbf{B}' \mathbf{B} \frac{W_{xx}}{2} + \left( \Theta' \Theta - r \right) MW_M \\
\quad + \left( a - \mathbf{B}' \Theta \right) W_x + \frac{\partial W}{\partial t} + G (\lambda M_t, t) - rW
\]

(66)

which is solved subject to the boundary condition

\[ W (M_T, x_T, T) = G_B (\lambda M_T, T). \]
Solution for Consumption

- Either equation (60) or (66) leads to the solution 
  $$W(M_t, x_t, t; \lambda) = W_t$$ 
  that determines $$\lambda$$ as a function of $$W_t$$, $$M_t$$, and $$x_t$$.

- The solution for $$\lambda$$ is then be substituted into (58) and (59) to obtain $$C_s^*(M_s)$$ and $$W_T^*(M_T)$$.

- When the individual follows this optimal policy, it is time consistent in the sense that should the individual resolve the optimal consumption problem at some future date, say, $$s > t$$, the computed value of $$\lambda$$ will be the same as that derived at date $$t$$. 
Portfolio Allocation

- Market completeness permits replication of the individual’s optimal process for wealth and its consumption dividend.
- The individual’s wealth follows the process

\[
dW = \omega' (\mu - re) W dt + (rW - C_t) dt + W \omega' \Sigma dZ \quad (67)
\]

where \( \omega = (\omega_1 \ldots \omega_n)' \) are portfolio weights and \( \mu = (\mu_1 \ldots \mu_n)' \) are assets’ expected rates of return.
- Equating the coefficients of wealth’s Brownian motions in (67) and (62) implies \( W \omega' \Sigma = \Sigma'_W \).
- Substituting in (64) for \( \Sigma_W \) and rearranging:

\[
\omega = -\frac{MW}{W} \Theta + \frac{W}{W} \Sigma'^{-1} B \quad (68)
\]
Optimal Portfolio Weights

- The no-arbitrage condition (53) in matrix form is
\[ \mu - re = \Sigma \Theta \]  
(69)

- Using (69) to substitute for \( \Theta \), equation (68) is
\[
\omega = - \frac{MW_M}{W} \Sigma^{-1} \Sigma'^{-1} (\mu - re) + \frac{W_x}{W} \Sigma'^{-1} B \\
= - \frac{MW_M}{W} \Omega^{-1} (\mu - re) + \frac{W_x}{W} \Sigma'^{-1} B 
\]  
(70)

- A comparison to (38) for the case of perfect correlation between assets and state variables shows that \( MW_M = J_W/J_{WW} \) and \( W_x = -J_{Wx}/J_{WW} \).

- Given \( W(M, x, t) \) in (60) or (66), the solution is complete.
Example of Wachter JFQA (2002)

- Let there be a risk-free asset with constant rate of return $r > 0$, and a single risky asset with price process

$$\frac{dS}{S} = \mu(t) \, dt + \sigma \, dz$$  \hspace{1cm} (71)

- Volatility, $\sigma$, is constant but the market price of risk, $\theta(t) = [\mu(t) - r] / \sigma$, satisfies the Ornstein-Uhlenbeck process

$$d\theta = a (\bar{\theta} - \theta) \, dt - b dz$$  \hspace{1cm} (72)

where $a$, $\bar{\theta}$, and $b$ are positive constants.

- Since $\mu(t) = r + \theta(t) \, \sigma$ so that $d\mu = \sigma \, d\theta$, the expected rate of return is lower (higher) after its realized return has been high (low).
Individual’s Expected Utility

- With CRRA and a zero bequest, (55) is

\[
\max_{C_s \forall s \in [t, T]} E_t \left[ \int_t^T e^{-\rho s} \frac{C^\gamma}{\gamma^\gamma} ds \right] + \lambda \left( M_t W_t - E_t \left[ \int_t^T M_s C_s ds \right] \right)
\]  
(73)

- The first-order condition (58) is

\[
C_s^* = e^{-\frac{\rho s}{1-\gamma}} (\lambda M_s)^{-\frac{1}{1-\gamma}}, \quad \forall s \in [t, T]
\]  
(74)

so that (60) is

\[
W_t = E_t \left[ \int_t^T \frac{M_s}{M_t} e^{-\frac{\rho s}{1-\gamma}} (\lambda M_s)^{-\frac{1}{1-\gamma}} ds \right]
\]  
(75)
Wealth and the Pricing Kernel

- $E_t \left[ M_s^{-\gamma / (1-\gamma)} \right]$ could be computed by noting that
  
  \[ \frac{dM}{M} = -rdt - \theta dz \]
  
  and $\theta$ follows the process in (72).

- Alternatively, $W_t$ can be solved using PDE (66):

  \[
  0 = \frac{1}{2} \theta^2 M^2 W_{MM} + \theta b M W_{M\theta} + \frac{1}{2} b^2 W_{\theta\theta} + (\theta^2 - r) MW_M \\
  + [a (\bar{\theta} - \theta) + b \theta] W_{\theta} + \frac{\partial W}{\partial t} + e^{-\frac{\rho t}{1-\gamma}} (\lambda M_t)^{-\frac{1}{1-\gamma}} - r W \\
  \]

  subject to boundary condition $W(M_T, \theta_T, T) = 0$. 
When $\gamma < 0$, so the individual is more risk averse than log utility, the solution to (76) is

$$W_t = (\lambda M_t)^{-\frac{1}{1-\gamma}} e^{-\frac{\rho t}{1-\gamma}} \int_0^{T-t} H(\theta_t, \tau) \, d\tau$$

(77)

where $H(\theta_t, \tau)$ is the exponential of a quadratic function of $\theta_t$ given by

$$H(\theta_t, \tau) \equiv e^{\frac{1}{1-\gamma} \left[ A_1(\tau) \frac{\theta_t^2}{2} + A_2(\tau) \theta_t + A_3(\tau) \right]}$$

(78)
and where

\[ A_1 (\tau) = \frac{2c_1 (1 - e^{-c_3 \tau})}{2c_3 - (c_2 + c_3)(1 - e^{-c_3 \tau})} \]

\[ A_2 (\tau) = \frac{4c_1 a\theta (1 - e^{-c_3 \tau/2})^2}{c_3 [2c_3 - (c_2 + c_3)(1 - e^{-c_3 \tau})]} \]

\[ A_3 (\tau) = \int_0^\tau \left[ \frac{b^2 A_2^2 (s)}{2 (1 - \gamma)} + \frac{b^2 A_1 (s)}{2} + a\theta A_2 (s) + \gamma r - \rho \right] ds \]

with \( c_1 \equiv \gamma / (1 - \gamma) \), \( c_2 \equiv -2 (a + c_1 b) \), and

\( c_3 \equiv \sqrt{c_2^2 - 4c_1 b^2} / (1 - \gamma) \).
Equation (77) can be inverted to solve for $\lambda$, but since from (74) $(\lambda M_t)^{-\frac{1}{1-\gamma}} e^{-\frac{\rho t}{1-\gamma}} = C^*_t$, (77) can be rewritten

$$C^*_t = \frac{W_t}{\int_0^{T-t} H(\theta_t, \tau) \, d\tau}$$

Note that wealth equals the value of consumption from now until $T - t$ periods into the future.

Therefore, since $\int_0^{T-t} H(\theta_t, \tau) \, d\tau = W_t / C^*_t$, the function $H(\theta_t, \tau)$ equals the value of consumption $\tau$ periods in the future scaled by current consumption.
Consumption Implications

- When \( \gamma < 0 \) and \( \theta_t > 0 \), so that \( \mu(t) - r > 0 \), then
  \[ \frac{\partial (C_t^*/W_t)}{\partial \theta_t} > 0; \]
  that is, the individual consumes a greater proportion of wealth the larger is the risky asset’s excess rate of return.

- This is what one expects given our earlier analysis showing that the "income" effect dominates the "substitution" effect when risk aversion is greater than that of log utility.
Portfolio Choice

- The weight (70) for a single risky asset is

\[
\omega = - \frac{MW}{W} \frac{\mu(t) - r}{\sigma^2} - \frac{W_\theta \ b}{W \ \sigma} \tag{80}
\]

- Using (77), \(-MW/M = 1/(1-\gamma)\) and \(W_\theta\) can be computed. Substituting these two derivatives into (80) gives

\[
\omega = \frac{\mu(t) - r}{(1-\gamma) \sigma^2} - \frac{b}{(1-\gamma) \sigma} \int_0^{T-t} H(\theta_t, \tau) \left[ A_1(\tau) \theta_t + A_2(\tau) \right] d\tau \\
= \frac{\mu(t) - r}{(1-\gamma) \sigma^2} - \frac{b}{(1-\gamma) \sigma} \int_0^{T-t} \frac{H(\theta_t, \tau)}{\int_0^{T-t} H(\theta_t, \tau) \ d\tau} \left[ A_1(\tau) \theta_t + A_2(\tau) \right] d\tau \tag{81}
\]
Portfolio Implications

- The first term of (81) is the mean-variance efficient portfolio.
- The second term is the hedging demand.
- $A_1(\tau)$ and $A_2(\tau)$ are negative when $\gamma < 0$, so that if $\theta_t > 0$, the term $[A_1(\tau)\theta_t + A_2(\tau)]$ is unambiguously negative and, therefore, the hedging demand is positive.
- Hence, individuals more risk averse than log invest more wealth in the risky asset than if investment opportunities were constant.
- Because of negative correlation between risky-asset returns and future investment opportunities, overweighting in the risky asset means that unexpectedly good returns today hedge against returns that are expected to be poorer tomorrow.
Summary

- We considered an individual’s continuous-time consumption and portfolio choice problem when asset returns followed diffusion processes.
- With constant investment opportunities, asset returns are lognormally distributed and optimal portfolio weights are similar to those of the single-period mean-variance model.
- With changing investment opportunities, optimal portfolio weights reflect demand components that seek to hedge against changing investment opportunities.
- The Martingale Approach to solving for an individual’s optimal consumption and portfolio choices is applicable to a complete markets setting where asset returns can perfectly hedge against changes in investment opportunities.