We consider two types of lifetime utility functions that are not time separable: habit persistence and recursive utility.

Habit persistence utility allows past consumption to play a role in determining current utility.

Two examples are the “internal” habit model of Constantinides (1990) and the “external” habit model of Campbell and Cochrane (1999).

Recursive utility makes current utility depend on expected values of future utility, and we study a continuous-time version of the model by Obstfeld (1994).
Constantinides’ Internal Habit Model Assumptions

- Constantinides’ internal habit formation model derives a representative individual’s consumption and portfolio choices in a simple production economy.

- **Technology**: A single capital-consumption good can be invested in up to two different technologies. The first is a risk-free technology whose output, $B_t$, follows the process

  $$dB/B = r \, dt$$  

  (1)

- The second is a risky technology whose output, $\eta_t$, satisfies

  $$d\eta/\eta = \mu \, dt + \sigma \, dz$$  

  (2)

  where $r$, $\mu$, and $\sigma$ are constants so there are constant investment opportunities.
Assumptions (continued)

- **Preferences**: Representative agents have date $t$ consumption of $C_t$ and maximize

\[
E_0 \left[ \int_0^\infty e^{-\rho t} u \left( \hat{C}_t \right) \, dt \right]
\]

where $u \left( \hat{C}_t \right) = \hat{C}_t^{\gamma} / \gamma$, $\gamma < 1$, $\hat{C}_t = C_t - bx_t$, and

\[
x_t \equiv e^{-at} x_0 + \int_0^t e^{-a(t-s)} C_s \, ds
\]

- $x_t$ is an exponentially weighted sum of past consumption.
- $b = 0$ is time-separable constant relative risk aversion utility, while $b < 0$ implies “consumption durability.”
- When $b > 0$, $bx_t$ is “subsistence” or “habit” consumption with $\hat{C}_t = C_t - bx_t$ referred to as “surplus” consumption.
Additional Parametric Assumptions

- Let $W_0$ be the initial wealth of the representative individual.
- The additional parametric assumptions are made:

\begin{align*}
W_0 &> \frac{b x_0}{r + a - b} > 0 \\
(r + a) &> b > 0 \\
\rho - \gamma r - \frac{\gamma (\mu - r)^2}{2(1 - \gamma)\sigma^2} &> 0 \\
0 &\leq m \equiv \frac{\mu - r}{(1 - \gamma)\sigma^2} \leq 1
\end{align*}
Reasons for Assumptions

- Conditions (5) and (6) ensure that an admissible (feasible) consumption and portfolio choice strategy exists that enables $C_t > b x_t$.

- To see this, note that the individual’s wealth dynamics are

$$dW = \left\{[(\mu - r)\omega_t + r]W - C_t\right\} dt + \sigma \omega_t W dz \quad (9)$$

where the risky technology weight satisfies $0 \leq \omega_t \leq 1$.

- Now if $\omega_t = 0$ for all $t$ and consumption equals a fixed proportion of wealth, $C_t = (r + a - b) W_t$, then

$$dW = \left\{rW - (r + a - b)W\right\} dt = (b - a) Wdt \quad (10)$$

- Equation (10) implies

$$W_t = W_0 e^{(b-a)t} > 0 \quad (11)$$
This implies \( C_t = (r + a - b) W_0 e^{(b-a)t} > 0 \) and

\[
C_t - bx_t = (r + a - b) W_0 e^{(b-a)t} - b \left[ e^{-at} x_0 + \int_0^t e^{-a(t-s)} (r + a - b) W_0 e^{(b-a)s} ds \right]
\]

\[
= (r + a - b) W_0 e^{(b-a)t} - \left[ e^{-at} bx_0 + b(r + a - b) W_0 e^{-at} \int_0^t e^{bs} ds \right]
\]

\[
= (r + a - b) W_0 e^{(b-a)t} - \left[ e^{-at} bx_0 + (r + a - b) W_0 e^{-at}(e^{bt} - 1) \right]
\]

\[
= e^{-at} \left[ (r + a - b) W_0 - bx_0 \right]
\]

which is greater than zero by assumption (5).
Reasons for Assumptions

- Condition (7) is a transversality condition that ensures that if the individual follows an optimal policy, the expected utility of consumption over an infinite horizon is finite.

- Condition (8) ensures that the individual chooses a nonnegative amount of wealth in the risky and risk-free technologies.

- Note that \( m \equiv \frac{\mu - r}{(1 - \gamma)\sigma^2} \) is the optimal risky-asset portfolio weight for the time-separable, constant relative-risk-aversion case.
The individual’s maximization problem is

$$\max_{\{C_s, \omega_s\}} E_t \left[ \int_t^\infty e^{-\rho s} \frac{[C_s - bx_s]^{\gamma}}{\gamma} ds \right] \equiv e^{-\rho t} J(W_t, x_t) \quad (13)$$

subject to (4) and (9).

Given the infinite horizon, we can simplify the indirect utility function

$$\hat{J}(W_t, x_t, t) = e^{-\rho t} J(W_t, x_t).$$

Note from (4) that the dynamics of $x(t)$ are:

$$\frac{dx}{dt} = -ae^{-at}x_0 + C_t - a \int_0^t e^{-a(t-s)} C_s \, ds,$$

or (14)

$$dx = (C_t - ax_t) \, dt \quad (15)$$
The Bellman equation is then

\[
0 = \max_{\{C_t, \omega_t\}} \left\{ U(C_t, x_t, t) + L[e^{-\rho t} J] \right\}
\]

\[
= \max_{\{C_t, \omega_t\}} \left\{ e^{-\rho t} \gamma^{-1} (C_t - bx_t)^\gamma \
+ e^{-\rho t} J_W [((\mu - r)\omega_t + r)W - C_t] \
+ \frac{1}{2} e^{-\rho t} J_{WW} \sigma^2 \omega_t^2 W^2 \
+ e^{-\rho t} J_x (C_t - ax_t) - \rho e^{-\rho t} J \right\}
\]

(16)
First Order Conditions

- The first-order condition with respect to $C_t$ is

$$\left( C_t - bx_t \right)^{\gamma-1} = J_W - J_x, \quad \text{or}$$

$$C_t = bx_t + [J_W - J_x]^{\frac{1}{\gamma-1}} \quad \text{(17)}$$

- The first-order condition with respect to $\omega_t$ is

$$\left( \mu - r \right) WJ_W + \omega_t \sigma^2 W^2 J_{WW} = 0, \quad \text{or}$$

$$\omega_t = -\frac{J_W}{J_{WW} W} \frac{\mu - r}{\sigma^2} \quad \text{(18)}$$
Equilibrium Partial Differential Equation

- Substituting (17) and (18) back into (16):

\[
0 = \frac{1 - \gamma}{\gamma} [J_W - J_x]^{\gamma-1} - \frac{J_W^2}{J_{WW}} \frac{(\mu - r)^2}{2\sigma^2} \\
+ (rW - bx)J_W + (b - a)xJ_x - \rho J
\]  

(19)

- When \( a = b = x = 0 \), we saw that \( J(W) = kW^\gamma \), so that \( u = C^\gamma/\gamma, u_c = J_W \), and

\[
C_t^* = (\gamma k)^{\frac{1}{\gamma-1}} W_t = W_t \left[ \rho - r\gamma - \frac{1}{2} \left( \frac{\gamma}{1 - \gamma} \right) \frac{(\mu - r)^2}{\sigma^2} \right] / (1-\gamma)
\]

(20)

and

\[
\omega_t^* = m
\]

(21)
Solution for Derived Utility of Wealth

For the time-inseparable case, we try the form

\[ J(W, x) = k_0 [W + k_1 x]^\gamma \]  \hspace{1cm} (22)

Substituting into (19) and setting the coefficients on \( x \) and \( W \) equal to zero, we find

\[ k_0 = \frac{(r + a - b) h^{\gamma^{-1}}}{(r + a)^{\gamma}} \]  \hspace{1cm} (23)

where

\[ h \equiv \frac{r + a - b}{(r + a)(1 - \gamma)} \left[ \rho - \gamma r - \frac{\gamma (\mu - r)^2}{2(1 - \gamma) \sigma^2} \right] > 0 \]  \hspace{1cm} (24)

and

\[ k_1 = -\frac{b}{r + a - b} < 0. \]  \hspace{1cm} (25)
Given the solution for \( J \), (17) and (18) imply

\[
C^*_t = bx_t + h \left( W_t - \frac{bx_t}{r + a - b} \right)
\]

(26)

and

\[
\omega^*_t = m \left[ 1 - \frac{bx_t/W_t}{r + a - b} \right]
\]

(27)

Since \( r + a > b \), so that \( \omega^*_t < m \), agents invest less in the risky asset and wealth has lower volatility compared to the time-separable case.
Dynamics of Consumption

- Consider the dynamics of the term \( W_t - \frac{b x_t}{r + a - b} \) in \( C_t^* \):

\[
d \left[ W_t - \frac{b x_t}{r + a - b} \right] = \left\{ \left[ (\mu - r) \omega_t^* + r \right] W_t - C_t^* \right\} dt + \sigma \omega_t^* W_t dz
\]

- Substituting in for \( \omega_t^* \) and \( C_t^* \), one obtains

\[
d \left[ W_t - \frac{b x_t}{r + a - b} \right] = \left[ W_t - \frac{b x_t}{r + a - b} \right] [ndt + m\sigma dz]
\]

where

\[
n \equiv \frac{r - \rho}{1 - \gamma} + \frac{(\mu - r)^2(2 - \gamma)}{2(1 - \gamma)^2 \sigma^2}
\]
Using (29) and (26), one can show

\[ \frac{dC_t}{C_t} = \left[ n + b - \frac{(n + a)bx_t}{C_t} \right] dt + \left( \frac{C_t - bx_t}{C_t} \right) m\sigma dz \]  \tag{31}

From the term \( \left( \frac{C_t - bx_t}{C_t} \right) m\sigma dz \), consumption growth is smoother than in the case of no habit persistence.

For a given equity (risky-asset) risk premium, this can imply a relatively smooth consumption path, even though risk aversion, \( \gamma \), may not be high in magnitude.

Recall the Hansen-Jagannathan bound for the time-separable case

\[ \left| \frac{\mu - r}{\sigma} \right| \leq (1 - \gamma) \sigma_c \]  \tag{32}
For the current habit persistence case, from (31):

\[
\sigma_{c,t} = \left( \frac{C_t - bx_t}{C_t} \right) m\sigma \tag{33}
\]

\[
= \left( \frac{\hat{C}_t}{C_t} \right) \left[ \frac{\mu - r}{(1 - \gamma)\sigma^2} \right] \sigma
\]

Define the *surplus consumption ratio* \( S_t \equiv \frac{\hat{C}_t}{C_t} \) and rearrange (33):

\[
\frac{\mu - r}{\sigma} = \frac{(1 - \gamma)\sigma_{c,t}}{S_t} \tag{34}
\]

Since \( S_t \equiv \frac{C_t - bx_t}{C_t} < 1 \) habit persistence may help reconcile the empirical violation of the H-J bound.
The Campbell-Cochrane External Habit Model

- This model has “keeping up with the Joneses” preferences and makes the following assumptions.

- **Technology**: There is a discrete-time endowment economy where date $t$ aggregate consumption-output, $C_t$, follows the lognormal process:

$$\ln (C_{t+1}) - \ln (C_t) = g + \nu_{t+1}$$  \hspace{1cm} (35)

where $\nu_{t+1} \sim N(0, \sigma^2)$ and is independently distributed.
Preferences

- **Preferences**: A representative individual maximizes

\[
E_0 \left[ \sum_{t=0}^{\infty} \delta^t \left( \frac{C_t - X_t}{\gamma} \right)^\gamma - 1 \right]
\]  

(36)

where \( \gamma < 1 \) and \( X_t \) denotes the “habit level” that is related to the surplus consumption ratio, \( S_t \equiv \frac{C_t - X_t}{C_t} \), where

\[
\ln \left( S_{t+1} \right) = (1 - \phi) \ln \left( \bar{S} \right) + \phi \ln \left( S_t \right) + \lambda \left( S_t \right) \nu_{t+1}
\]  

(37)

and where \( \lambda \left( S_t \right) \) is the *sensitivity function*

\[
\lambda \left( S_t \right) = \frac{1}{S} \sqrt{1 - 2 \left[ \ln \left( S_t \right) - \ln \left( \bar{S} \right) \right]} - 1
\]  

(38)

and

\[
\bar{S} = \sigma \sqrt{\frac{1 - \gamma}{1 - \phi}}
\]  

(39)
In Constantinides (1990) an individual’s habit depends on her own past consumption, so that when choosing $C_t$ she takes into account how it will affect her future utility.

In Campbell and Cochrane (1999) an individual’s habit depends on everyone else’s current and past consumption, so that when choosing $C_t$ she views $X_t$ as exogenous.

The external habit assumption simplifies the agent’s decision making because habit is an exogenous state variable that depends on aggregate, not the individual’s, consumption.
The individual’s marginal utility of consumption is

\[ u_c (C_t, X_t) = \left( C_t - X_t \right)^{\gamma^{-1}} = C_t^{\gamma^{-1}} S_t^{\gamma^{-1}} \quad (40) \]

and the representative agent’s stochastic discount factor is

\[ m_{t,t+1} = \delta \frac{u_c (C_{t+1}, X_{t+1})}{u_c (C_t, X_t)} = \delta \left( \frac{C_{t+1}}{C_t} \right)^{\gamma^{-1}} \left( \frac{S_{t+1}}{S_t} \right)^{\gamma^{-1}} \quad (41) \]
Let \( r = -\ln (E_t [m_{t,t+1}]) \) be the continuously compounded risk-free rate between dates \( t \) and \( t + 1 \):

\[
\begin{align*}
  r &= -\ln \left( \delta E_t \left[ e^{-(1-\gamma)\ln(C_{t+1}/C_t)-(1-\gamma)\ln(S_{t+1}/S_t)} \right] \right) \\
  &= -\ln \left( \delta e^{-(1-\gamma)E_t[\ln(C_{t+1}/C_t)]-(1-\gamma)E_t[\ln(S_{t+1}/S_t)]} \right) \\
  &\quad \times e^{\frac{1}{2}(1-\gamma)^2Var_t[\ln(C_{t+1}/C_t)+\ln(S_{t+1}/S_t)]} \\
  &= -\ln (\delta) + (1 - \gamma) g + (1 - \gamma) (1 - \phi) (\ln \bar{S} - \ln S_t) \\
  &\quad - \frac{1}{2} (1 - \gamma)^2 \sigma^2 [1 + \lambda (S_t)]^2
\end{align*}
\]

Substituting in for \( \lambda (S_t) \) shows that the rate is constant:

\[
r = -\ln (\delta) + (1 - \gamma) g - \frac{1}{2} (1 - \gamma) (1 - \phi)
\]
Price of Market Portfolio

- Aggregate consumption equals the economy’s aggregate dividends (output) paid by the market portfolio. Therefore,

\[ P_t = E_t [m_{t,t+1} (C_{t+1} + P_{t+1})] \quad (44) \]

- The price-dividend ratio for the market portfolio is:

\[
\frac{P_t}{C_t} = \delta E_t \left[ m_{t,t+1} \frac{C_{t+1}}{C_t} \left( 1 + \frac{P_{t+1}}{C_{t+1}} \right)^\gamma \left( 1 + \frac{P_{t+1}}{C_{t+1}} \right) \right] \quad (45)
\]
Solution

- Solve forward this difference equation by repeatedly updating and substituting for $P_{t+i}/C_{t+i}$ to obtain:

$$
rac{P_t}{C_t} = E_t \left[ \sum_{i=1}^{\infty} \delta^i \left( \frac{S_{t+i}}{S_t} \right)^{\gamma-1} \left( \frac{C_{t+i}}{C_t} \right)^\gamma \right]
$$

(46)

- The solution is computed numerically by simulating the lognormal processes for $C_t$ and $S_t$, noting that $S_{t+1}/S_t$ depends on the current level of $S_t$.

- $P_t/C_t$ varies only with $S_t$, so that the portfolio’s expected returns and volatility are also functions of $S_t$. 

Coefficient of Relative Risk Aversion

- Note that the coefficient of relative risk aversion is
  \[ -\frac{C_t u_{cc}}{u_c} = \frac{1 - \gamma}{S_t} \]  
  (47)

- As shown earlier, when consumption is lognormally distributed
  the H-J bound is approximately
  \[ \left| \frac{E[r_i] - r}{\sigma_{r_i}} \right| \leq -\frac{C_t u_{cc}}{u_c} \sigma_c = \frac{(1 - \gamma) \sigma_c}{S_t} \]  
  (48)

which is similar to Constantinides’ internal habit model except, here, \( \sigma_c \) is a constant and \( E[r_i] \) and \( \sigma_{r_i} \) are time-varying functions of \( S_t \).
The coefficient of relative risk aversion is relatively high when $S_t$ is relatively low, such as during a recession.

Moreover, the model predicts that the equity risk premium increases during a recession (when $-\frac{C_t u_{cc}}{u_c}$ is high), which seems to be a phenomenon of the postwar U.S. stock market.

Campbell and Cochrane calibrate the model to U.S. consumption and stock market data and, due to the nonlinear form for $S_t$, have more success in describing actual asset returns.
Recursive Utility

- Recursive utility is forward looking, and was developed by Kreps and Porteus (1978) and Epstein and Zin (1989).
- We will follow Duffie and Epstein (1992) and study the continuous-time limit of recursive utility.
- Recall that time-separable utility takes the form

\[ V_t = E_t \left[ \int_t^T U(C_s, s) \, ds \right] \]  

(49)

where \( U(C_s, s) \) is often specified \( U(C_s, s) = e^{-\rho(s-t)}u(C_s) \).

- Recursive utility, however, takes the form

\[ V_t = E_t \left[ \int_t^T f(C_s, V_s) \, ds \right] \]  

(50)

where \( f \) is known as an *aggregator function*. 
Features of Recursive Utility

- Utility (50) is recursive since current lifetime utility, $V_t$, depends on expected values of future lifetime utility, $V_s$, $s > t$.
- When $f$ has appropriate properties, Duffie and Epstein (1992) show that a Bellman-type equation can be used to derive optimal consumption and portfolio choices.
- We consider a form of recursive utility that generalizes power (CRRA) utility.
- Unlike CRRA where the elasticity of intertemporal substitution, $\epsilon$, must equal the inverse of the coefficient of relative risk aversion, $1/(1 - \gamma)$, recursive utility distinguishes $\epsilon$ (an intertemporal consumption-savings choice concept) from $(1 - \gamma)$ (an atemporal asset risk choice concept).
Assumptions of the Obstfeld Model

- **Technology**: There is a production economy where a capital-consumption good can be invested in two different technologies. The first is a risk-free technology whose output, $B_t$, follows the process

  $$\frac{dB}{B} = r dt$$  \hspace{1cm} (51)

  The second is a risky technology whose output, $\eta_t$, follows the process

  $$\frac{d\eta}{\eta} = \mu dt + \sigma dz$$  \hspace{1cm} (52)

  Since $r$, $\mu$, and $\sigma$ are constants, there are constant investment opportunities.
Recursive Preferences

Preferences: Representative, infinitely-lived agents maximize

\[ V_t = E_t \int_t^\infty f(C_s, V_s) \, ds \]  \hspace{1cm} (53)

where \( f \), the aggregator function, is given by

\[ f(C_s, V_s) = \rho \frac{C_s^{1-\frac{1}{\epsilon}} - [\gamma V_s]^{\frac{\epsilon-1}{\epsilon\gamma}}}{(1 - \frac{1}{\epsilon}) [\gamma V_s]^{\frac{\epsilon-1}{\epsilon\gamma}} - 1} \]  \hspace{1cm} (54)

\( \rho > 0 \) is the agent’s rate of time preference; \( \epsilon > 0 \) is the elasticity of intertemporal substitution; and \( 1 - \gamma > 0 \) is the coefficient of relative risk aversion. When \( \epsilon = 1/(1 - \gamma) \), (53) and (54) are (ordinally) equivalent to

\[ V_t = E_t \int_t^\infty e^{-\rho s} \frac{C_s^{\gamma}}{\gamma} \, ds \]  \hspace{1cm} (55)
Derived Utility of Wealth

- If $\omega_t$ is the weight invested in the risky asset (technology), wealth satisfies

$$dW = [\omega(\mu - r)W + rW - C] \, dt + \omega \sigma W \, dz \quad (56)$$

- Define $J(W_t)$ as the maximized lifetime utility at date $t$:

$$J(W_t) = \max_{\{C_s, \omega_s\}} E_t \int_t^\infty f(C_s, V_s) \, ds \quad (57)$$

$$= \max_{\{C_s, \omega_s\}} E_t \int_t^\infty f(C_s, J(W_s)) \, ds$$

- Due to the infinite horizon problem with constant investment opportunities, $f(C, V)$ is not an explicit function of calendar time and the only state variable is $W$. 
The Bellman equation is

$$0 = \max_{\{C_t, \omega_t\}} f [C_t, J(W_t)] + L [J(W_t)]$$  \hspace{1cm} (58)$$

or

$$0 = \max_{\{C_t, \omega_t\}} f [C, J(W)] + J_W [\omega (\mu - r) W + rW - C]$$  \hspace{1cm} (59)$$

$$+ \frac{1}{2} J_{WW} \omega^2 \sigma^2 W^2$$
First-Order Conditions

- The first-order condition with respect to $C_t$ is

$$\rho \frac{C^{-\frac{1}{\epsilon}}}{[\gamma J]^\frac{\epsilon-1}{\epsilon\gamma}-1} - J_W = 0 \quad (60)$$

or

$$C = \left(\frac{J_W}{\rho}\right)^{-\epsilon} [\gamma J]^\frac{1-\epsilon}{\gamma} + \epsilon \quad (61)$$

- The first-order condition with respect to $\omega_t$ is

$$J_W (\mu - r) W + J_{WW} \omega \sigma^2 W^2 = 0 \quad (62)$$

or

$$\omega = -\frac{J_W}{J_{WW} W} \frac{\mu - r}{\sigma^2} \quad (63)$$
Substituting the optimal values for \( C \) and \( \omega \) into (59):

\[
\frac{J_W}{\rho} \left( \frac{J_W}{\rho} \right)^{1-\epsilon} \left[ \gamma J \right]^{(\epsilon-1) \left[ 1 - \frac{\epsilon-1}{\epsilon \gamma} \right]} - \left[ \gamma J \right]^{\frac{1-\epsilon}{\epsilon \gamma}} \\
\frac{\rho}{(1 - \frac{1}{\epsilon})} \left[ \gamma J \right]^{rac{\epsilon-1}{\epsilon \gamma} - 1} \\
+ J_W \left[ - \frac{J_W}{J_{WW}} \frac{(\mu - r)^2}{\sigma^2} + rW - \left( \frac{J_W}{\rho} \right)^{-\epsilon} \left[ \gamma J \right]^{\frac{1-\epsilon}{\epsilon \gamma} + \epsilon} \right] \\
+ \frac{1}{2} \frac{J_W^2}{J_{WW}} \frac{(\mu - r)^2}{\sigma^2} = 0
\]
Simplifying, one obtains:

\[
\frac{\epsilon \rho}{\epsilon - 1} \left[ \left( \frac{J_W}{\rho} \right)^{-\epsilon} \left[ \frac{1}{\gamma} (1 - \epsilon) + \epsilon - \frac{1}{\gamma} J \right] ight]
\]

\[
+ J_W \left[ - \frac{J_W}{J_{WW}} \frac{(\mu - r)^2}{\sigma^2} + rW - \left( \frac{J_W}{\rho} \right)^{-\epsilon} \left[ \frac{1}{\gamma} (1 - \epsilon) + \epsilon \right] \right]
\]

\[
+ \frac{1}{2} \frac{J_{WW}^2}{J_{WW}} \frac{(\mu - r)^2}{\sigma^2} = 0
\]
Guessing a solution of the form $J(W) = (aW)^{\gamma} / \gamma$ and substituting into (65), one finds that $a = \alpha^{1/(1-\epsilon)}$ where

$$\alpha \equiv \rho^{-\epsilon} \left( \epsilon \rho + (1 - \epsilon) \left[ r + \frac{(\mu - r)^2}{2(1 - \gamma) \sigma^2} \right] \right)$$ \hspace{1cm} (66)

In turn, substituting this value for $J$ into (61), one obtains

$$C = \alpha \rho^\epsilon W$$ \hspace{1cm} (67)

$$= \left( \epsilon \rho + (1 - \epsilon) \left[ r + \frac{(\mu - r)^2}{2(1 - \gamma) \sigma^2} \right] \right) W$$

and the optimal portfolio weight of the risky asset is

$$\omega = \frac{\mu - r}{(1 - \gamma) \sigma^2}$$ \hspace{1cm} (68)
Bhamra and Uppal (2003) show that when investment opportunities are stochastic, the portfolio weight, $\omega$, can depend on both $\gamma$ and $\epsilon$.

Note when $\epsilon = 1 / (1 - \gamma)$, equation (67) equals

$$C = \frac{\gamma}{1-\gamma} \left[ \frac{\rho}{\gamma} - r - \frac{(\mu-r)^2}{2(1-\gamma)\sigma^2} \right] W$$

derived earlier for the CRRA case.

For an infinite horizon solution to exist, $C_t$ in (67) must be positive, requiring

$$\rho > \frac{\epsilon-1}{\epsilon} \left( r + [\mu - r]^2 / [2 (1 - \gamma) \sigma^2] \right),$$

which occurs when $\epsilon$ is small.

For example, when $\rho > 0$, the inequality is satisfied when $\epsilon < 1$. 
For $C^*$ in (67), the term $r + [\mu - r]^2 / [2 (1 - \gamma) \sigma^2]$ can be rewritten by substituting $\omega = (\mu - r) / [(1 - \gamma) \sigma^2]$:

$$r + \frac{(\mu - r)^2}{2 (1 - \gamma) \sigma^2} = r + \omega \frac{\mu - r}{2} \quad (69)$$

An increase in (69) raises (reduces) $C$ when $\epsilon < 1$ ($\epsilon > 1$).

The intuition is that when $\epsilon < 1$, the income effect from an improvement in investment opportunities dominates the substitution effect, so that consumption rises and savings fall.

The reverse occurs when $\epsilon > 1$: the substitution effect dominates the income effect and savings rise.
Assuming $0 < \omega < 1$ and substituting (67) and (68) into (56), wealth follows the geometric Brownian motion:

$$\frac{dW}{W} = [\omega^* (\mu - r) + r - \alpha \rho^e] \, dt + \omega^* \sigma \, dz$$

(70)

$$= \left[ \frac{(\mu - r)^2}{(1 - \gamma) \sigma^2} + r - \epsilon \rho - (1 - \epsilon) \left( r + \frac{(\mu - r)^2}{2 (1 - \gamma) \sigma^2} \right) \right] \, dt$$

$$+ \frac{\mu - r}{(1 - \gamma) \sigma} \, dz$$

$$= \left[ \epsilon \left( r + \frac{(\mu - r)^2}{2 (1 - \gamma) \sigma^2} - \rho \right) + \frac{(\mu - r)^2}{2 (1 - \gamma) \sigma^2} \right] \, dt$$

$$+ \frac{\mu - r}{(1 - \gamma) \sigma} \, dz$$
Note that since $C = \alpha \rho^e W$, then $dC/C$ has the same drift and volatility as wealth in (70).

Thus, $d \ln C$ has a volatility, $\sigma_c$, and a mean, $g_c$, equal to

$$\sigma_c = \frac{\mu - r}{(1 - \gamma) \sigma}$$  \hspace{1cm} (71)$$

and

$$g_c = \epsilon \left( r + \frac{(\mu - r)^2}{2(1 - \gamma) \sigma^2} - \rho \right) + \frac{(\mu - r)^2}{2(1 - \gamma) \sigma^2} - \frac{1}{2} \sigma_c^2$$

$$= \epsilon \left( r + \frac{(\mu - r)^2}{2(1 - \gamma) \sigma^2} - \rho \right) - \frac{\gamma (\mu - r)^2}{2(1 - \gamma)^2 \sigma^2}$$  \hspace{1cm} (72)$$
Comparative Statics

- From (72), if \( r + [\mu - r]^2 / [2 (1 - \gamma) \sigma^2] > \rho \), growth rises with \( \epsilon \) as individuals save more.

- The squared Sharpe ratio, \( [\mu - r]^2 / \sigma^2 \) is a measure of the attractiveness of the risky asset, and the sign of \( \partial g_c / \partial ( [\mu - r]^2 / \sigma^2 ) \) equals the sign of \( \epsilon - \gamma / (1 - \gamma) \).

- For the CRRA case of \( \epsilon = 1 / (1 - \gamma) \), the derivative is positive, so that \( \partial g_c / \partial \mu > 0 \) and \( \partial g_c / \partial \sigma < 0 \).

- In general, \( \partial g_c / \partial ( [\mu - r]^2 / \sigma^2 ) < 0 \) if \( \epsilon < \gamma / (1 - \gamma) \) since from (68) as agents invest more in the faster-growing risky asset they also raise \( C_t \) (and reduce savings) when \( \epsilon < 1 \).

- Thus, when \( \epsilon < \gamma / (1 - \gamma) \), less savings dominates the portfolio effect and the economy grows more slowly.
Obstfeld points out that the integration of global financial markets that allows residents to hold risky foreign, as well as domestic, investments increases diversification and effectively reduces individuals’ portfolio variance, $\sigma^2$.

The model predicts that if $\epsilon > \gamma/(1 - \gamma)$, financial market integration causes countries to grow faster.

This recursive utility model does not help in explaining the equity premium puzzle since, from (71), the risky-asset Sharpe ratio, $(\mu - r)/\sigma$, equals $(1 - \gamma)\sigma_c$, the same form as with time-separable utility.
Recursive utility might explain the risk-free rate puzzle: substitute (71) into (72) and solve for $r$:

$$r = \rho + \frac{g_c}{\epsilon} - \left[ 1 - \gamma - \frac{\gamma}{\epsilon} \right] \frac{\sigma_c^2}{2} \quad (73)$$

Recall that when $\epsilon = 1 / (1 - \gamma)$ we have

$$r = \rho + (1 - \gamma) g_c - (1 - \gamma)^2 \frac{\sigma_c^2}{2} \quad (74)$$

Empirically, $g_c \approx 0.018$ is large relative to $\sigma_c^2/2 \approx 0.03^2/2 = 0.00045$, so the net effect of higher risk aversion, $1 - \gamma$, needed to fit the equity risk premium implies too high a risk-free rate in (74).

(73) may circumvent this problem because $g_c$ is divided by $\epsilon$. 

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From (70) and (72), if the risky-asset Sharpe ratio, \( (\mu - r) / \sigma \), is independent of the level of the real interest rate, \( r \), then \( \epsilon \) can be estimated by regressing consumption growth, \( d \ln C \), on the real interest rate, \( r \).

Tests using aggregate consumption data find that \( \epsilon \) is small, often close to zero.

Other tests based on disaggregated consumption data find higher estimates for \( \epsilon \), often around 1.

A value of \( \epsilon = 1 \) makes \( r \) independent of \( \gamma \) and, assuming \( \rho \) is small, could produce a reasonable value for \( r \).
Summary

- For utility with habit persistence, the standard coefficient of relative risk aversion, $1 - \gamma$, is transformed to $(1 - \gamma) / S_t$ where $S_t < 1$ is the surplus consumption ratio.
- These models may imply aversion to holding risky assets sufficient to justify a high equity risk premium.
- Recursive utility distinguishes between an individual’s level of risk aversion and his elasticity of intertemporal substitution.
- Such utility might allow a high equity risk premium and a low risk-free interest rate that is present in historical data.